On the observer dependence of the Hilbert space near the horizon of black holes

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Abstract

One of the pronounced characteristics of gravity, distinct from other interactions, is that there are no local observables which are independent of the choice of the spacetime coordinates. This property acquires crucial importance in the quantum domain in that the structure of the Hilbert space pertinent to different observers can be drastically different. Such intriguing phenomena as the Hawking radiation and the Unruh effect are all rooted in this feature. As in these examples, the quantum effect due to such observer-dependence is most conspicuous in the presence of an event horizon and there are still many questions to be clarified in such a situation. In this paper, we attempt to perform a comprehensive study of the observer dependence of the quantum Hilbert space of a massless scalar field in the vicinity of the horizon of the Schwarzschild black holes in four dimensions, both in the eternal (two-sided) case and in the physical (one-sided) case created by collapsing matter. Specifically, we compare and relate the Hilbert spaces of the three types of observers, namely (i) the freely falling observer, (ii) the observer who stays at a fixed proper distance outside of the horizon and (iii) the natural observer inside of the horizon analytically continued from outside. The relations we obtain have a number of important implications, such as on the quantum equivalence principle and the related firewall phenomenon, on the number of degrees of freedom seen by each type of observer, and on the "thermal-type" spectrum of particles realized in a pure state.

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Contents

1 Introduction

A quantum black hole is a fascinating but as yet an abstruse object. Recent endeavors to identify it in a suitable class of CFTs in the AdS/CFT context [1–3] [4–7] or by an ingeneous model such as the one proposed by Sachdev-Ye-Kitaev [8–10] have seen only a glimpse of it, to say the most. Unfortunately, the string theory, at the present stage of development, does not seem to give us a useful clue either. This difficulty is naturally expected since an object whose profile fluctuates by quantum self-interaction would be hard to capture. We must continue our struggle to find an effective means to characterize it more precisely.

Although the quantization of a black hole itself is still a formidable task, some analyses of quantum effects around a (semi-)classical black hole have been perfomed since long time ago and they have already uncovered various intriguing phenomena. Among them are the celebrated Hawking radiation [11] [12–14] and closely related Unruh effect [15] [16, 33]. These effects revealed the non-trivial features of the quantization in curved spacetimes, in particular in those with event horizons. At the same time, they brought out new puzzles of deep nature, such as the problem of the information loss, the final fate of the evaporating black hole, and so on.

More recently, further unexpected quantum effect in the black hole environment was argued to occur, namely that a freely falling observer encounters excitations of high energy quanta, termed "firewall", as he/she crosses the event horizon of a black hole [17,18] [19]. This is clearly at odds with the equivalence principle, which is one of the foundations of classical general relativity. An enormous number of papers have appeared since then, both for and against the assertion[1] . Various arguments presented have all been rather indirect, however, making use of the properties of the entanglement entropy, application of the no cloning theorem, use of information theoretic arguments, etc..

At the bottom of these phenomena lies the strong dependence of the quantization on the frame of observers, which is one of the most characteristic features of quantum gravity. This is particularly crucial when the spacetime of interest contains event horizons as seen by some observers.

The main aim of the present work is to investigate this observer dependence in some physically important situations as explicitly as possible to gain some firm and direct understanding of the phenomena rooted in this feature. For this purpose, we shall study the quantization of a massless scalar field in the vicinity of the horizon of the Schwarzschild black hole in four dimensions as perfomed by three typical observers. They are (i) the

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[1]It is practically impossible to list all such papers on this subject. We refer the reader to those citing the the basic papers [17,18].

freely falling observer crossing the horizon, (ii) the stationary observer hovering at a fixed proper distance outside the horizon (*i.e.* the one under constant acceleration), and (iii) the natural analytically continued observer inside the horizon.

Such an investigation, we believe, will be important for at least two reasons. One is that we will deal directly with the states of the scalar fields as seen by different observers and will not rely on any indirect arguments alluded to above. This makes the interpretation of the outcome of our study quite transparent (up to certain approximations that we must make for computation). Another role of our investigation is that the concrete result we obtain should serve as the properties of quantum fields in the background of a black hole, which should be compared, in the semi-classical regime, to the results to be obtained by other means of investigation, notably and hopefully by the AdS/CFT duality[2]. For some progress and intriguing proposals in the related directions, see [20–27]. This is important since, as far as we are aware, there has not been a serious attempt to understand how the observer dependence is described in the context of AdS/CFT duality.

We will perform our study both for the case of two-sided eternal Schwarzschild black hole and for that of one-sided physical black hole modeled by a simple Vaidya metric produced by collapsing matter or radiation at the speed of light[3] [28–30]. What makes such an investigation feasible explicitly is the well-known fact that near the horizon of the Schwarzschild black hole (roughly within the Schwarzschild radius from the horizon; see Sec. 3.1 for more precise estimate) there exists a coordinate frame in which the metric takes the form of the flat four dimensional Miknowski spacetime $M^{1,3}$. Thus, one can make use of the knowledge of the quantization in the flat space for observers corresponding to the various Rindler frames. As this will serve as the platform upon which we develop our picture and computational methods for the black hole cases, we will give, in Sec. 2, a review of this knowledge together with some further new information about the relations between the quantizations by the three aforementioned observers.

In making use of this flat space approximation to the near horizon region of a black hole, an important care must be taken, however. Although the scalar field and its canonical conjugate momentum are locally well-approximated by those in the flat space for the region of our interest and hence the canonical quantization can be performed without any problem, as we try to extract the physical modes which create and annihilate the quantum states, such a local knowledge is not enough in general. This is because the notion of *a quantum state* requires the global information of the wave function. Technically, this is reflected in the fact that the orthogonality relation needed for the extraction of the mode

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[2]As far as the vicinity of the horizon is concerned, the Schwarzschild black hole and the AdS black hole have the same structure.

[3]Actually, we shall make an infinitesimal regularization to make the trajectory of the matter slightly timelike in order to avoid certain singularity.

is expressed by an integral over the entire spacelike surface at equal time, and depending on the region of interest such a surface may not be totally contained within the region where the flat space approximation is valid.

One such problem, which however can be easily dealt with, stems from the simple fact that the approximation by the four-dimensional flat space includes that of the spherical surface of the horizon by a tangential plane around a point. Clearly since the physical modes of the scalar field should better be classified by the angular momentum, not by the linear momentum, we shall use $\mathbb{R}^{1,1} \times \mathbb{S}^2$, instead of $M^{1,3}$, as the more accurately approximated spacetime, where $\mathbb{R}^{1,1}$ stands for a portion of two-dimensional flat spacetime realized near the horizon and $\mathbb{S}^2$ is the sphere at the Schwarzschild radius. Various formulas reviewed and/or developed in Sec. 2 for $M^{1,3}$ can be readily transplanted to this case by replacing the plane waves by the spherical harmonics. As for the treatment of the near horizon portion of $\mathbb{R}^{1,1}$, we shall devise some arguments to obtain essential features of the modes near the horizon in the radial direction.

Although we cannot summarize here all the results on how the different observers see their quanta and how they are related, let us list two, which are of obvious interest:

- As far as our study indicates the equivalence principle still holds quantum mechanically near the horizon of the black hole and the freely falling observer finds no surprize as he/she goes through the horizon.

- For the physical (one-sided) black hole, the vacuum[4] $|\hat{0}\rangle_-$ for the freely falling observer is a pure state which is not the same as the usual Minkowski vacuum $|0\rangle_M$. Nevertheless the expectation value of the number operator for the observer in the frame of the right Rindler wedge in $|\hat{0}\rangle_-$ has a Unruh-like distribution, which contains a "thermal" factor together with another portion depending on the assumed interaction between the scalar field and the collapsing matter, which is effectively expressed as a boundary condition. This is in contrast to the case of the two-sided eternal black hole, where tracing out of the modes of the left Rindler wedge must be performed and the resultant mixed state density matrix produces the usual purely thermal form of the Unruh distribution. The effect for the physical black hole occuring in the pure state described above is essentially of the same origin as the Hawking radiation seen by the asymptotic observer, who is a Rindler observer[5].

The plan of the rest of the paper is as follows: In Sec. 2, we begin by describing the quantization of a massless scalar field in four dimensional flat Minkowski space from the

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[4]The vacuum referred to here will be explained in Sec. 4.2.3.
[5]For related work, though in a different setting, see [32].

point of view of various observers and provide explicit relations beween them. Although this section is mostly a review, we derive some useful relations as well, which have not been explicitly discussed in the literature. This includes the construction of the explicit unitary transformation between the Minkowski mode operators and those of the future Rindler wedge and how the Poincaré algebra is realized in various wedges. Next in Sec. 3, this knowledge about the quantization in flat spacetime will be utilzed to discuss how the scalar field is quantized by various observers in the vicinity of the event horizon of a two-sided Schwarzschild black hole, which by a suitable choice of coordinates can be approximated by a part of $\mathbb{R}^{1,1}$ times $\mathbb{S}^2$. In Sec. 4, we study the similar problem in the case of a Vaidya model of a physical one-sided black hole which is produced by a collapse of null matter. The effect of this collapse is treated as an effective boundary condition on the scalar field along a regularized almost lightlike trajectory of the matter. Even though we focus on the flat region near the horizon, the quantum states, which depend on the global situation, show different properties as compared with the two-sided case studied in Sec. 3. In Sec. 5, we disucss the implications of the results obtained in the previous sections on some important questions, such as the quantum equivalence principle, the firewall phenomenon, and the Unruh effect near the horizon. Finally, in Sec. 6, after summarizing the results, we re-emphasize that the effect of the observer dependence of quantization is one of the most crucial characteristics of any theory of grantum gravity and it should be seriously investigated, in particular, in the framework of AdS/CFT approach. Several appendices are provided to give further useful details of the formulas and calculations discussed in the main text.

2   Quantization of a scalar field in the Rindler wedges and the degenerate Kasner universes

We begin by describing the quantization of a massless[6] scalar field in the four dimensional Minkowski space, from the standpoint of a uniformly accelerated Rindler observers for the right and the left wedges W_R and W_L, and their appropriate analytic continuations for the future and the past wedges W_F and W_P, which can be identified as degenerate Kasner universes. In Figure 2.1, we draw the trajectories of the corresponding observers and the equal time slices in each wedge.

The subject of the quantization by Rindler observers has a long history [33–36] and hence the content of this section is largely a review[7]. However, a part of our exposition supplements the description in the existing literature by providing some clarifying details

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[6]Massive case can be treated in an entirely similar manner.
[7]For a review article closely related to this section, see [37].
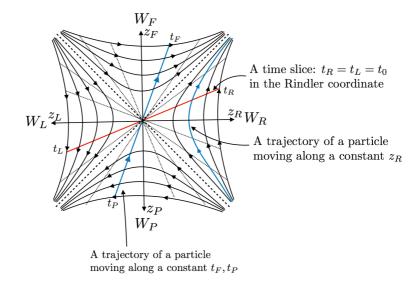
Figure 2.1: Trajectories and equal-time slices of the Rindler observers in various wedges. The boundaries of the wedges W_R, W_F, W_L and W_P are shown by dotted lines. The arrowed blue lines represent the trajectories of a particle, while the red line is a typical time slice at $t_R = t_L$ for W_R and W_L.

and new relations. The results of this section will serve as the foundation upon which to discuss the observer-dependent quantization around the horizon of Schwarzschild black holes, both eternal (two-sided) and physical (one-sided), as will be performed in Sec. 3.

2.1   Relation between the Minkowski and the Rindler coordinates

Before getting to the quantization of a scalar field, we need to describe the relationship between the Minkowski coordinate and the Rindler coordinates in various wedges.

The d-dimensional Minkowski metric is described in the usual Cartesian coordinate as

$$ds^2 = -(dt_M)^2 + (dx^1)^2 + \sum_{i=2}^{d-1}(dx^i)^2 \,. \tag{2.1}$$

Since we will be mostly concerned with the first two coordinates and the roles of the rest of the $d-2$ coordinates are essentially the same, hereafter we will deal with the four dimensional case, *i.e.* $d = 4$.

As for the Rindler coordinates, we begin with the one in the right wedge W_R shown in Figure 2.1. As is well-known, it is related to the coordinates of the observer who is acclerated in the positive x^1 direction with a uniform acceleration. The trajectory of the observer in the (t_M, x^1) Minkowski plane with a value of acceleration $\kappa(> 0)$ is given by

$$(x^1)^2 - (t_M)^2 = (1/\kappa)^2 = z_R^2 \,. \tag{2.2}$$

Here the symbol z_R is introduced as a variable, meaning that different values of z_R describes different trajectories. Thus the Rindler coordinate system is spanned by the proper time τ_R of the observer and the spatial coordinate z_R. The relation to the Minkowski coordinate is given by

$$t_M = z_R \sinh t_R \,, \qquad x^1 = z_R \cosh t_R \,, \qquad (z_R > 0) \,, \tag{2.3}$$

where we introduced for convenience the rescaled time t_R defined by

$$t_R \equiv \kappa \tau_R \,. \tag{2.4}$$

The metric in terms of these variables is

$$ds^2 = -z_R^2 dt_R^2 + dz_R^2 + \sum_{i=2}^{3} (dx^i)^2 \,. \tag{2.5}$$

Note that $z_R = 0$ corresponds to the (Rindler) horizon, which consists of two dimensional planes along the lightlike lines bounding the region W_R . It will often be convenient to use the following lightcone variables:

$$x^\pm \equiv x^1 \pm t_M = z_R e^{\pm t_R} \,. \tag{2.6}$$

This shows that t_R is nothing but the *rapidity-like variable* and gets simply translated by the Lorentz boost in the x^1 direction.

The coordinates (t_L, z_L) in the left wedge W_L can be obtained in an entirely similar manner and are related to the Minkowski coordinates by

$$t_M = -z_L \sinh t_L \,, \qquad x^1 = -z_L \cosh t_L \,, \qquad (z_L > 0) \,. \tag{2.7}$$

The metric takes exactly the same form as (2.5), with the subscript R replaced by L. Note that as t_R increases from $-\infty$ to ∞, the Minkowski time t_M also increases, while when t_L increases from $-\infty$ to ∞, t_M decreases, as indicated by the arrows in Figure 2.1.

Next consider the future and the past wedges, W_F and W_P. They describe the interior of the Rindler horizon. The relation to the Minkowski coordinate for W_F is

$$t_M = z_F \cosh t_F \,, \qquad x^1 = z_F \sinh t_F \,, \qquad (z_F > 0) \,, \tag{2.8}$$

and the metric takes the form

$$ds^2 = -dz_F^2 + z_F^2 dt_F^2 + \sum_{i=2}^{3} (dx^i)^2 \,. \tag{2.9}$$

This means that in W_F, z_F is the timelike and t_F is the spacelike coordinates. As in the case of W_R, the following lightcone combinations are often useful:

$$x^\pm \equiv x^1 \pm t_M = \pm z_F e^{\pm t_F} \,. \tag{2.10}$$

Just like t_R, under a Lorentz transformation the variable t_F undergoes a simple shift.

This interchange of the timelike and the spacelike natures also occurs in the past wedge W_P . In the entirely similar manner, we have

$$t_M = -z_P \cosh t_P\,, \qquad x^1 = -z_P \sinh t_P\,, \qquad (z_P > 0)\,, \qquad (2.11)$$

with the form of the metric identical to (2.9) with the subscript $F \to P$.

In Sec. 3, where we discuss how the similar Rindler wedges for a flat space appear in the vicinity of the horizon of a Schwarzschild black hole, we will see that the z_R variable expresses the proper distance from the horizon in the outside region and is related to the radial variable r and the Schwarzschild radius $2M$ (where M is the mass of the black hole), by $z_R \simeq \sqrt{8M(r - 2M)}$. Hence, as we go through the horizon from W_R into W_F, we must make an analytic continuation by choosing a branch for the square-root cut. Similarly, an analytic continutation connects W_F and W_L, and so on. Such a continuation process must be such that as we go once around all the wedges, we should come back to the same branch for W_R. A simple analysis for this consistency yields the following continutation rules, with a sign $\eta = \pm 1$ which can be chosen by convention, for the adjacent wedges:

$$t_F = t_R - i\frac{\pi}{2}\eta\,, \qquad z_F = e^{i(\pi/2)\eta} z_R\,, \qquad (2.12)$$

$$t_L = t_F - i\frac{\pi}{2}\eta\,, \qquad z_L = e^{-i(\pi/2)\eta} z_F = z_R\,, \qquad (2.13)$$

$$t_P = t_L + i\frac{\pi}{2}\eta = t_F\,, \qquad z_P = e^{-i(\pi/2)\eta} z_L\,, \qquad (2.14)$$

$$t_R = t_P + i\frac{\pi}{2}\eta\,, \qquad z_R = e^{i(\pi/2)\eta} z_P\,, \qquad (2.15)$$

$$z_R, z_F, z_L, z_P \geq 0\,. \qquad (2.16)$$

One can easily check that these relations are compatible with the relations between the Minkowski variables and the Rindler wedge variables given above.

2.2 Quantization in the Minkowski spacetime

We now discuss the quantization of a massless scalar field ϕ in various coordinates.

In this subsection, just for setting the notation, we summarize the simple case for the Minkowski coordinate. The action, the canonical momentum and the equation of motion

are given by

$$S = -\frac{1}{2} \int dt_M dx^1 d^2x \left(-(\partial_{t_M}\phi^M)^2 + (\partial_{x^1}\phi^M)^2 + \sum_{i=2}^{3}(\partial_{x^i}\phi^M)^2 \right) , \tag{2.17}$$

$$\pi^M \equiv \frac{\partial\mathcal{L}}{\partial(\partial_{t_M}\phi^M)} = \partial_{t_M}\phi^M , \tag{2.18}$$

$$\left(-\partial_{t_M}^2 + \partial_{x^1}^2 + \sum_{i=2}^{3}\partial_{x^i}^2 \right)\phi^M = 0 , \tag{2.19}$$

where we denote the fields and the time in the Minkowski frame with the super(sub)script M. Now ϕ^M can be expanded into Fourier modes as

$$\phi^M(t_M, x_M^1, x) = \int \frac{dp^1}{\sqrt{2\pi}\sqrt{2E_{kp^1}}} \int \frac{d^2k}{2\pi} e^{ikx+ip^1x^1 - iE_{kp^1}t_M} a_{kp^1}^M + \text{h.c.} , \tag{2.20}$$

$$E_{kp^1} \equiv \sqrt{(k)^2 + (p^1)^2} . \tag{2.21}$$

Here and throughout, we often denote (x^2, x^3) simply by x and similarly for the momenta for the corresponding dimensions by k, and write the inner product $\sum_{i=2}^{3}k_i x_i$ as kx. Canonical quantization is performed by demanding[8]

$$[\pi^M(t_M, x^1, x), \phi^M(t_M, y^1, y)] = -i\delta(x^1 - y^1)\delta(x - y) . \tag{2.22}$$

Using the orthogonality of the exponential function, we can easily extract out the mode operators and check that they satisfy the usual commutation relations:

$$[a_{kp^1}^M, a_{k'p'^1}^{M\dagger}] = \delta(p^1 - p'^1)\delta(k - k') , \qquad \text{rest} = 0 . \tag{2.23}$$

2.3 Quantization outside the Rindler horizon

Let us now begin the discussion of quantization in the Rindler coordinates in various wedges.

We first consider the Rindler wedges outside the horizon, namely W_R and W_L. Since the metrics in these wedges take the same form in the respective variables, we will focus on W_R. The action takes the form

$$S = -\frac{1}{2} \int dt_R dz_R d^2x \sqrt{-g} g^{\mu\nu} \partial_\mu\phi^R \partial_\nu\phi^R$$

$$= -\frac{1}{2} \int dt_R dz_R d^2x \left(-\frac{1}{z_R}(\partial_{t_R}\phi^R)^2 + z_R(\partial_{z_R}\phi^R)^2 + z_R\sum_{i=2}^{3}(\partial_{x^i}\phi^R)^2 \right) . \tag{2.24}$$

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[8]$\delta(x-y)$ of course means the two-dimensional delta-function $\delta^2(x-y)$. This abbreviation will be used throughout.

The canonical momentum is given by

$$\pi^R \equiv \frac{\partial \mathcal{L}}{\partial(\partial_{t_M} \phi^R)} = \frac{1}{z_R} \partial_{t_R} \phi^R \,, \tag{2.25}$$

which has an extra factor of $1/z_R$ compared with the Minkowski case. Variation of the action yields the equation of motion

$$\left(\partial_{z_R}^2 + \frac{1}{z_R} \partial_{z_R} + \sum_{i=2}^{3} \partial_{x^i}^2 - \frac{1}{z_R^2} \partial_{t_R}^2 \right) \phi^R = 0. \tag{2.26}$$

As it is a second order differential equation, there are two independent solutions, which can be taken to be the exponential function times the modified Bessel functions, namely $e^{i(kx-\omega t)} I_{i\omega}(|k|z)$ and $e^{i(kx-\omega t)} K_{i\omega}(|k|z)$. The appropriate solution is the one which damps at $z \to \infty$ and we write it as

$$f_{k\omega}^R(t_R, z_R, x) = N_\omega^R K_{i\omega}(|k|z_R) e^{i(kx-\omega t_R)} \,, \tag{2.27}$$

where N_ω is a normalization constant given below. Thus, the scalar field in the right Rindler wedge can be expanded as

$$\phi^R(t_R, z_R, x) = \int_0^\infty d\omega \int d^2 k N_\omega^R \left[K_{i\omega}(|k|z_R) e^{i(kx-\omega t_R)} a_{k\omega}^R + \text{h.c.} \right] \,,$$
$$N_\omega^R = \frac{\sqrt{\sinh \pi\omega}}{2\pi^2} \,. \tag{2.28}$$

Let us make some remarks on this formula:

(i) For the hermitian conjugate part, only the conjugation for the exponential part is needed since $K_{i\omega}(|k|z)$ is real.

(ii) The normalization constant chosen here will lead to the canonical form of the commutation relations, as explained in Appendix B.1.1.

(iii) The variable ω is the energy conjugate to the time-like variable t_R, and hence its range is $\omega \geq 0$.

Canonical quantization is performed by imposing the following equal-time commutation relation:

$$[\pi^R(t_R, z_R, x), \phi^R(t_R, z_R', x'),] = -i\delta(z_R - z_R')\delta(x - x') \,. \tag{2.29}$$

Using the orthogonality relation for the modified Bessel functions explained in the Appendix A.1, it is straightforward to obtain the commutation relations for the mode operators

$$[a_{\omega k}^R, a_{\omega' k'}^{R\dagger}] = \delta(\omega - \omega')\delta(k - k') \,, \qquad \text{rest} = 0 \,. \tag{2.30}$$

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For some details of the calculations, see Appendix B.1.1.

The quantization in W_L is essentially similar to the one in W_R above, except for one point that one must be careful about. Recall that as the Minkowski time t_M (and also t_R) goes from $-\infty$ to ∞, the time t_L in W_L runs oppositely from ∞ to $-\infty$. This is due to the definition of t_L by a smooth analytic continuation and does not of course mean that a physical particle moves from the future to the past. After all W_L is a part of the Minkowski space and all the particles and waves must evolve along the positive direction in Minkowski time. This applies to the W_L observer as well, who is under constant acceleration in the negative x^1 direction. The time which increases along the trajectory of the W_L observer is *not* t_L but $\tilde{t}_L \equiv -t_L$. Therefore, the quantization in this frame should be done with $\tilde{t}_L$ regarded as time. Then, all the formulas for the quantization in the W_R frame hold for the W_L frame, with t_R replaced by $\tilde{t}_L$. This means that *if one wishes to use the "time" t_L to write the mode expansion of the field ϕ^L and define its conjugate momentum π^L*, we have

$$\phi^L(t_L, z_L, x) = \int_0^\infty d\omega \int d^2 k N_\omega^L \left[K_{i\omega}(|k|z_L) e^{ikx + i\omega t_L} a_{k\omega}^L + \text{h.c.} \right] ,$$
$$N_\omega^L = \frac{\sqrt{\sinh \pi\omega}}{2\pi^2} , \qquad (2.31)$$

and

$$\pi^L(t_L, z_L, x) \equiv -\frac{1}{z_L} \partial_{t_L} \phi^L . \qquad (2.32)$$

One can then check that the equal time commutation relation $\left[\pi^L(t_L, z_L, x), \phi^L(t_L, z_L', x') \right] = -i\delta(z_L - z_L')\delta(x - x')$ holds correctly.

2.4   Quantization inside the Rindler horizon

Next consider the quantization in the Rindler wedges inside the horizon, *i.e.* in W_F and W_P. Again they can be treated in parallel and we focus on W_F. Compared to the previous analysis for the outside region, an important difference arises due to the interchange of the timelike and the spacelike coordinates.

The action in W_F region is given by

$$S = -\frac{1}{2} \int dt_F dz_F d^2x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi^F \partial_\nu \phi^F$$
$$= -\frac{1}{2} \int dt_F dz_F d^2x \left(-z_F(\partial_{z_F}\phi^F)^2 + \frac{1}{z_F}(\partial_{t_F}\phi^F)^2 + z_F \sum_{i=2}^3 (\partial_{x^i}\phi^F)^2 \right) . \qquad (2.33)$$

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From the sign of various terms, it is clear that t_F *is the space coordinate and z_F is the time coordinate.* Therefore, the canonical momentum must be defined by

$$\pi^F \equiv \frac{\partial \mathcal{L}}{\partial(\partial_{z_F}\phi^F)} = z_F \partial_{z_F}\phi^F. \tag{2.34}$$

The equation of motion takes the form

$$\left(\partial_{z_F}^2 + \frac{1}{z_F}\partial_{z_F} - \sum_i \partial_i^2 - \frac{1}{z_F^2}\partial_{t_F}^2\right)\phi^F = 0. \tag{2.35}$$

Again there are two independent solutions, which can be taken as

$$f_{k\omega}(t_F, z_F, x) = N_\omega^F H_{i\omega}^{(2)}(|k|z_F)e^{i(kx - \omega t_F)},$$
$$f_{k\omega}^*(t_F, z_F, x) = N_\omega^F e^{-\pi\omega} H_{i\omega}^{(1)}(|k|z_F)e^{-i(kx - \omega t_F)}. \tag{2.36}$$

where $H_{i\omega}^{(1)}$ and $H_{i\omega}^{(2)}$ are the Hankel functions of imaginary order and N_ω^F is the normalization constant, to be specified shortly.

To expand the scalar field in terms of these functions, a care should be taken as to which function should be associated with the annihilation (resp. creation) modes. This is because, in contrast to the previous case, ω is conjugate to the spacelike variable t_F and hence it is not the energy but the usual momentum. Therefore the range of ω is $-\infty \le \omega \le \infty$ and we cannot determine the positive (resp. negative) frequency mode from the exponential part of the functions above.

Thus, to determine which Hankel function should be taken as describing the positive frequency part, one must look at the asymptotic behavior of $H_{i\omega}^{(1,2)}(|k|z_F)$ at late time, *i.e.* at very large positive z_F. Such behaviors are given by

$$H_{i\omega}^{(1)}(|k|z_F) \sim e^{i|k|z_F - i\pi/4}e^{\pi\omega/2}\sqrt{\frac{2}{\pi|k|z_F}}, \tag{2.37}$$

$$H_{i\omega}^{(2)}(|k|z_F) \sim e^{-i|k|z_F + i\pi/4}e^{-\pi\omega/2}\sqrt{\frac{2}{\pi|k|z_F}}. \tag{2.38}$$

We see that $H_{i\omega}^{(2)}(|k|z_F)$ behaves like $e^{-i|k|z_F}$ which corresponds to the positive frequency with respect to "energy" $|k|$ (with an overall inessential damping behavior). This tells us that the the correct expansion is

$$\phi^F(t_F, z_F, x) = \int_{-\infty}^{\infty} d\omega \int d^2k N_\omega^F \left[e^{i(kx - \omega t_F)}H_{i\omega}^{(2)}(|k|z_F)a_{k\omega}^F + \text{h.c.}\right],$$
$$(N_\omega^F)^2 = \frac{e^{\pi\omega}}{8(2\pi)^2}, \tag{2.39}$$

where the factor N_ω^F is determined such that the commutator of the modes take the canonical form as in (2.41) below. In the literature the modes $a_{k\omega}^F$ are often called the Unruh modes, whereas the modes $a_{k\omega}^R$ are referred to as the Rindler modes.

Now the canonical quantization is performed by imposing the "equal-time" (*i.e.* equal z_F) commutation relation relation

$$[\pi^F(z_F, t_F, x), \phi^F(z_F, t'_F, x'),] = -i\delta(t_F - t'_F)\delta(x - x')\,. \tag{2.40}$$

Using the orthogonality of the Hankel functions, we get the canonical form of the commutation relations for creation/ annihilation operators, namely,

$$[a_{k\omega}^F, a_{k'\omega'}^{F\dagger}] = \delta(\omega - \omega')\delta(k - k')\,, \qquad \text{rest} = 0\,. \tag{2.41}$$

See Appendix B.1.2 for some details of this computation.

2.5   Hamiltonian in the future wedge

We have seen that in W_F and W_P the timelike and the spacelike variables are swapped compared to the usual situations in W_R and W_L and this has made the identification of the positive and negative frequency modes somewhat non-trivial. In fact, this swapping makes the Hamiltonian in W_F and W_P *time-dependent*. In this subsection, we briefly discuss the form of the Hamiltonian and its action as the proper time-development operator.

From the action (2.33) for the W_F region, the Hamiltonian is readily obtained as

$$H_F = \frac{1}{2}\int dt_F\left(\frac{1}{z_F}(\pi^F)^2 + \frac{1}{z_F}(\partial_{t_F}\phi^F)^2 + z_F(\sum_{i=2}^3 \partial_{x^i}\phi^F)^2\right)\,, \qquad \pi^F = z_F\partial_{z_F}\phi^F\,. \tag{2.42}$$

Since z_F is the time variable, the Hamiltonian H_F is clearly time-dependent. Therefore the time-development of a state $|\psi(z_F)\rangle$ is accomplished by the unitary operator $U(z_F)$ in the manner

$$|\psi(z_F)\rangle = U(z_F)|\psi(0)\rangle\,, \tag{2.43}$$

$$U(z_F) = T\exp\left(-i\int_0^{z_F} H_F(z')dz'\right)\,, \tag{2.44}$$

where $T\exp(**)$ denotes the time-ordered exponential. Thus for general z_F the time-development is quite non-trivial.

We now wish to express H_F in terms of modes given in (2.39) and see how it simplifies for large z_F. The necessary computation is straightforward: Substitute the expansion (2.39) and perform the space integral over t_F. Since the intermediate expressions are

lengthy, we omit them and display the final form. It is given by

$$
\begin{aligned}
H_F = \frac{\pi}{8} \int_{-\infty}^{\infty} d\omega d^2k & \left[\left(\frac{\omega^2}{z_F} + z_F k^2 \right) H_{i\omega}^{(2)}(|k|z_F) H_{-i\omega}^{(2)}(|k|z_F) \right. \\
& \left. + z_F \partial_{z_F} H_{i\omega}^{(2)}(|k|z_F) \partial_{z_F} H_{-i\omega}^{(2)}(|k|z_F) \right] a_{k\omega}^F a_{-k,-\omega}^F + \text{h.c.} \\
+ \frac{\pi}{4} \int_{-\infty}^{\infty} d\omega d^2k & \left[\left(\frac{\omega^2}{z_F} + z_F k^2 \right) H_{i\omega}^{(2)}(|k|z_F) H_{i\omega}^{(1)}(|k|z_F) \right. \\
& \left. + z_F \partial_{z_F} H_{i\omega}^{(2)}(|k|z_F) \partial_{z_F} H_{i\omega}^{(1)}(|k|z_F) \right] a_{k\omega}^{F\dagger} a_{k\omega}^F ,
\end{aligned}
\tag{2.45}
$$

where we have discarded, as usual, an infinite constant coming from the normal ordering of the last term.

Now let us consider the limit of large time, $z_F \to \infty$. In this limit, since $t_M = \sqrt{z_F^2 + (x^1)^2}$, the line of equal time will approach that of equal Minkowski time t_M and hence we expect that H_F will take the form for the free scalar field. Using the formulas (2.37) and (2.38) for large z, we can drastically simplify the expressions for H_F. The leading term which does not vanish as $z_F \to \infty$ takes the form

$$
H_F\big|_{z \to \infty} = \int_{-\infty}^{\infty} d\omega \, d^2k \, |k| a_{k\omega}^{F\dagger} a_{k\omega}^F .
\tag{2.46}
$$

This is z_F-independent and indeed coincides with the form for the free scalar field in Minkowski space.

2.6 Relation between the quantizations in W_R, W_L, W_F and the Minkowski frames

We are ready to discuss the relation between the quantizations in W_R, W_F and the Minkowski frames.

2.6.1 Minkowski and W_R frames

First, since W_R is contained in the Minkowski space, it should be possible to express the modes in the W_R frame in terms of the modes in the Minkowski frame. Using the Klein-Gordon inner product for W_R defined in Appendix B.1, we obtain the expression for the annihilation operator $a_{k\omega}^R$ in the W_R frame as

$$
\begin{aligned}
a_{k\omega}^R &= (f_{k\omega}^R, \phi^M)_{\text{KG}}^R \\
&= i \int_0^{\infty} \frac{dz_R}{z_R} \int dx^2 (f_{k\omega}^{R*} \overleftrightarrow{\partial_{t_R}} \phi^M) \\
&= \int_{-\infty}^{\infty} \frac{dp^1}{\sqrt{4\pi E_{kp^1}}} \frac{1}{\sqrt{\sinh \pi\omega}} \left(\frac{E_{kp^1} - p^1}{E_{kp^1} + p^1} \right)^{\frac{-i\omega}{2}} \left[e^{\pi\omega/2} a_{kp^1}^M + e^{-\pi\omega/2} a_{-kp^1}^{M\dagger} \right] ,
\end{aligned}
\tag{2.47}
$$

where $\omega \geq 0$. Some details of the calculations are given in Appendix B.2.

Actually, this expression for $a_{k\omega}^R$ can be simplified rather drastically by introducing the rapidity variable u defined by

$$u \equiv \frac{1}{2} \ln \left(\frac{E_{kp^1} + p^1}{E_{kp^1} - p^1} \right) . \tag{2.48}$$

Then we can immediately solve this relation for E_{kp^1} and p^1 in terms of u and obtain

$$E_{kp^1} = |k| \cosh u , \qquad p^1 = |k| \sinh u . \tag{2.49}$$

Furthermore, the integration measures are related as

$$dp^1 = |k| \cosh u\, du = E_{kp^1} du , \tag{2.50}$$

with the identical range of integration $[-\infty, \infty]$ for both p^1 and u. Further, if we define the annihilation operator in the rapidity variable as

$$a_{ku}^M \equiv \sqrt{|k| \cosh u}\, a_{kp^1}^M = \sqrt{E_{kp^1}}\, a_{kp^1}^M , \tag{2.51}$$

the commutation relation with its conjugate is

$$[a_{ku}^M, a_{k'u'}^{M\dagger}] = |k| \sqrt{\cosh u \cosh u'} \delta(p^1 - p'^1) \delta(k - k') = \delta(u - u') \delta(k - k') , \tag{2.52}$$

where we used $\delta(p^1 - p'^1) = \delta(|k| \sinh u - |k| \sinh u') = \delta(u - u')/(|k| \cosh u)$.

Using these definitions, the relation (2.47) can be written as

$$a_{k\omega}^R = \int_{-\infty}^{\infty} \frac{du}{\sqrt{4\pi \sinh \pi\omega}} e^{i\omega u} \left[e^{\pi\omega/2} a_{ku}^M + e^{-\pi\omega/2} a_{-ku}^{M\dagger} \right] . \tag{2.53}$$

Note that, as is well-known, the annihilation operator $a_{k\omega}^R$ is composed both of the annihilation and the creation operators of the Minkowski frame. Another important fact is that there is no negative frequency modes, $a_{k\omega}^R$(for $\omega < 0$), in the W_R frame and consequently it is *not* possible to invert the relation above to express the Minkowski annihilation/creation operators in terms of the ones in the W_R frame only. This means that *the degrees of freedome that W_R observer sees is half as many as those seen by the Minkowski observer.*

2.6.2 Minkowski and W_F frames related by a Fourier transform

The situation is different for the quantization in the W_F frame. By using the Klein-Gordon inner product for W_F, we can obtain the relation between the annihilation operator $a_{\omega k}^F$ in the W_F frame and the mode operators in the Minkowski frame. This time, what we obtain is the relation

$$a_{k\omega}^F = (f_{k\omega}^F, \phi^M)_{\text{KG}}^F = i \int_{-\infty}^{\infty} z dt_F dx^2 (f_{k\omega}^{F*} \overleftrightarrow{\partial_{z_F}} \phi^M)$$

$$= i \int_{-\infty}^{\infty} \frac{dp^1}{\sqrt{2\pi E_{kp^1}}} \left(\frac{E_{kp^1} - p^1}{E_{kp^1} + p^1} \right)^{-\frac{i\omega}{2}} a_{kp^1}^M , \tag{2.54}$$

which requires only the annihilation operator in the Minkowski frame. Furthermore, since ω is conjugate to the *spacelike* coodinate t_R in this case, we *do* have negative frequency modes for $a_{k\omega}^F$, $\omega < 0$ and hence *the number of degress of freedom of the modes that the W_{F} observer sees are the same as those for the Minkowski observer.*

As in the case of $a_{k\omega}^R$, the relation (2.54) above can be simplified by the use of the rapidity variable u. It can be written as

$$a_{k\omega}^F = i \int \frac{du}{\sqrt{2\pi}} e^{i\omega u} a_{ku}^M . \tag{2.55}$$

Apart from a factor of i, this is nothing but the Fourier transformation. Therefore the inverse relation is trivial to obtain and we get

$$a_{ku}^M = -i \int \frac{d\omega}{\sqrt{2\pi}} e^{-i\omega u} a_{k\omega}^F , \tag{2.56}$$

$$\Leftrightarrow \quad a_{kp^1}^M = -i \int \frac{d\omega}{\sqrt{2\pi E_{k'p^1}}} \left(\frac{E_{kp^1} - p^1}{E_{kp^1} + p^1} \right)^{\frac{i\omega}{2}} a_{k\omega}^F . \tag{2.57}$$

The fact that $a_{kp^1}^M$ and $a_{k\omega}^F$ are in one to one correspodence with no mixing of the creation and the annihilation operators tells us that the vacuum state of the two observers are the same, namely[9]

$$|0\rangle_M = |0\rangle_F . \tag{2.58}$$

The important difference, however, is that the entities recognized as "particles" by the two observers are quite distinct and their wave functions have "dual" profiles.

2.6.3 Fourier transform as a unitary transformation

We now make a useful observation that the Fourier transform exhibited above can be realized by a unitary transformation, in the sense to be described below. For some references, see [35, 38].

Define the fourier transform $\tilde{g}(p)$ of a function $g(x)$ as

$$\int \frac{dx}{\sqrt{2\pi}} e^{ipx} g(x) = \tilde{g}(p) . \tag{2.59}$$

The *functional forms* of $g(x)$ and $\tilde{g}(p)$ are in general different.

Let us look for a special class of functions for which the functional forms of their Fourier transform are the same up to a proportionality constant. The simplest such function is obviously the following Gaussian for which the proportionality constant is unity:

$$f_0(x) \equiv \pi^{-1/4} e^{-x^2/2} , \qquad \tilde{f}_0(p) = \pi^{-1/4} e^{-p^2/2} = f_0(p^2) . \tag{2.60}$$

---

[9]The vacuum $|0\rangle_F$ is called the "Unruh vacuum".

We know that such a function is the coordinate representation of the ground state of the one-dimensional harmonic oscillators $\{a, a^\dagger\}$

$$f_0(x) = \langle x|0 \rangle \,, \tag{2.61}$$

where $|0\rangle$ denotes the oscillator ground state defined by

$$a|0\rangle = 0 \,, \qquad \left[a, a^\dagger\right] = 1 \,, \qquad \langle 0|0\rangle = 1 \,. \tag{2.62}$$

and $|x\rangle$ is, as usual, the eigenstate of the operator $\hat{x}$ with the eigenvalue x, i.e. $\hat{x}|x\rangle = x|x\rangle$.

In what follows, we take the coordinate representations of a and $a^\dagger$ as

$$a = \frac{1}{\sqrt{2}}(x + ip) = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right) = \frac{i}{\sqrt{2}}\left(\frac{d}{dp} + p\right) \,, \tag{2.63}$$

$$a^\dagger = \frac{1}{\sqrt{2}}(x - ip) = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right) = \frac{(-i)}{\sqrt{2}}\left(-\frac{d}{dp} + p\right) \,. \tag{2.64}$$

Now, as is well-known, the x-representation of the excited states of the oscillator system

$$|n\rangle \equiv \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \,, \qquad \langle m|n\rangle = \delta_{m,n} \tag{2.65}$$

is given by

$$f_n(x) \equiv \langle x|n\rangle = \frac{1}{\sqrt{n!}}\left[\frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right)\right]^n \langle x|0\rangle = \frac{1}{\sqrt{n!}}\left[\frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right)\right]^n f_0(x) \,. \tag{2.66}$$

Inserting the unity $\int (dp/\sqrt{2\pi})|p\rangle\langle p|$ and using $\langle x|p\rangle = e^{ipx}$, this can be written as the Fourier transform

$$f_n(x) = \langle x|n\rangle = \int \frac{dp}{\sqrt{2\pi}}\langle x|p\rangle\langle p|n\rangle = \int \frac{dp}{\sqrt{2\pi}}\langle x|p\rangle(-i)^n \frac{1}{\sqrt{n!}}\left[\frac{1}{\sqrt{2}}\left(-\frac{d}{dp} + p\right)\right]^n \langle p|0\rangle$$

$$= \int \frac{dp}{\sqrt{2\pi}}e^{ixp}(-i)^n f_n(p) \,. \tag{2.67}$$

Thus the functional form of the Fourier transform $\tilde{f}_n(p)$ is the same as the original up to a constant, namely $\tilde{f}_n(p) = (-i)^n f_n(p)$.

Let us consider the number operator $\mathcal{N} = a^\dagger a$, for which $\mathcal{N}|n\rangle = n|n\rangle$. By using the p-representation of a and $a^\dagger$, as exhibited in (2.63) and (2.64), this is written as

$$\mathcal{N}_p f_n(p) = \frac{1}{2}\left(-\frac{d^2}{dp^2} + p^2 - 1\right) f_n(p) = n f_n(p) \,. \tag{2.68}$$

Therefore, we can express the Fourier transform $(-i)^n f_n(p)$ as

$$e^{-i\frac{\pi}{2}\mathcal{N}_p} f_n(p) = (-i)^n f_n(p) \,. \tag{2.69}$$

Note that here the terminology "Fourier transform" refers to the transform of the *form of the function*, with the argument taken to be the same.

Exactly the same formulas hold for p replaced by x. Thus as far as the set of functions $\{f_n(p)\}$ are concerned, the Fourier transform is realized by the operation on the LHS of (2.69).

Up to a constant, $f_n(x)$ is nothing but the Hermite polynomial $H_n(x)$ times the Gaussian $e^{-x^2/2}$. More precisely,

$$f_n(x) = \frac{1}{(2^n n! \sqrt{\pi})^{1/2}} H_n(x) e^{-x^2/2} , \tag{2.70}$$

where $H_n(x)$ is defind by[10]

$$H_n(x) \equiv e^{x^2/2} \left(-\frac{d}{dx} + x \right)^n e^{-x^2/2} . \tag{2.71}$$

Now in order to apply this formalism to the oscillators, such as a_{ku}^M and $a_{k\omega}^F$, we consider a set of oscillators depending on a continuous variable and satisfying the following commutation relations

$$\left[a(x), a^\dagger(y) \right] = \delta(x - y) . \tag{2.72}$$

Since so far we have realized the Fourier transform as a differential operation on the set of functions $f_n(x)$ only, in order to define the Fourier transform of such an oscillator function, we should first express $a(x)$ and $a^\dagger(x)$ in terms of $f_n(x)$. This can be done due to the following completeness relation

$$\delta(x - y) = \sum_{n=0}^{\infty} f_n(x) f_n(y) . \tag{2.73}$$

Thus, expanding

$$a(x) = \sum_{m=0}^{\infty} b_m f_m(x) , \qquad a^\dagger(y) = \sum_{n=0}^{\infty} b_n^\dagger f_n(y) , \tag{2.74}$$

the commutation relation can be reproduced as

$$\left[a(x), a^\dagger(y) \right] = \sum_{m,n} [b_m, b_n^\dagger] f_m(x) f_n(y) = \delta(x - y) , \tag{2.75}$$

provided we take $[b_m, b_n^\dagger] \equiv \delta_{m,n}$. Therefore, since the Fourier transform is a linear operation, we can apply the formula (2.69) to the operators $a(x)$ and $a^\dagger(x)$ as well. This can

---

[10]There are different conventions for the normalization of the Hermite polynomials. Our definition is the most standard one.

be implemented formally by the unitary transformation of the form

$$\tilde{a}(x) = U^\dagger a(x) U \,, \tag{2.76}$$

$$U = \exp\left(-\frac{i\pi}{2}\int dy\, a^\dagger(y)\mathcal{N}_y a(y)\right) \,. \tag{2.77}$$

In fact one can easily verify

$$U^\dagger a(x) U = a(x) + \left[\frac{i\pi}{2}\int dy\, a^\dagger(y)\mathcal{N}_y a(y), a(x)\right] + \cdots = e^{-\frac{i\pi}{2}\mathcal{N}_x} a(x)\,. \tag{2.78}$$

So indeed the Fourier transform for the form of the operator is reproduced.

Applied to the oscillators a_{ku}^M and $a_{k\omega}^F$, we have the relations

$$\begin{aligned}
ia_{ku}^M &= U_F a_{k\omega}^F U_F^\dagger\big|_{\omega=u}\,, \\
a_{k\omega}^F &= U_M^\dagger i a_{ku}^M U_F\big|_{u=\omega}\,,
\end{aligned} \tag{2.79}$$

where we defined

$$U_\mathcal{I} = \exp\left(\frac{i\pi}{2}\int d\omega'\, a_{k\omega'}^{\mathcal{I}\dagger}\mathcal{N}_{\omega'} a_{k\omega'}^{\mathcal{I}}\right)\,, \quad \mathcal{I} = F, M. \tag{2.80}$$

In using the operators $U_\mathcal{I}$, one must make sure to act the differential operator $\mathcal{N}_\omega$ on any ω-dependent quantity, be it a function or an operator, to the right of it. Transformations using $U_\mathcal{I}$ are useful in converting various quantities in the Minkowski and the W_F frames, as will be demonstrated for the Poincaré generators in Appendix C.

2.6.4 Relations between W_R, W_L, W_F and Minkowski frames

Finally, let us relate the modes in W_R and W_L frames with those in the W_F frame. Combining (2.47) and (2.54) (and their hermitian conjugates) and eliminating the Minkowski modes, we can obtain a simple *algebraic* relationship between $a_{k\omega}^R$ and $a_{k\omega}^F$:

$$a_{k\omega}^R = -\frac{i}{\sqrt{2\sinh\pi\omega}}\left[e^{\pi\omega/2}a_{k\omega}^F - e^{-\pi\omega/2}a_{-k,-\omega}^{F\dagger}\right]\,, \qquad \omega \geq 0\,. \tag{2.81}$$

Again since $a_{k\omega}^R$ exists only for $\omega \geq 0$, this relation cannot be inverted.

However, recall that the "full" Rindler spacetime has the left wedge W_L in addition to the right wedge W_R. By similar arguments we can obtain the relationship between $a_{k\omega}^L$ and $a_{k\omega}^F$ as

$$a_{k\omega}^L = -\frac{i}{\sqrt{2\sinh\pi\omega}}\left[e^{\pi\omega/2}a_{-k,-\omega}^F - e^{-\pi\omega/2}a_{k\omega}^{F\dagger}\right]\,, \qquad \omega \geq 0\,. \tag{2.82}$$

Note that the right hand side contains $a_{k,-\omega}^F$ instead of $a_{k\omega}^F$, in contrast to the expression of $a_{k\omega}^R$ given in (2.81). Therefore, combining (2.81) and (2.82), one can express the modes in the future wedge W_F in terms of the modes in W_R and W_L in the following combinations:

$$a_{k\omega}^F = -\frac{i}{\sqrt{2\sinh\pi\omega}}\left[e^{\pi\omega/2}a_{k\omega}^R - e^{-\pi\omega/2}a_{k\omega}^{L\dagger}\right]\,, \tag{2.83}$$

$$a_{-k,-\omega}^F = \frac{i}{\sqrt{2\sinh\pi\omega}}\left[e^{-\pi\omega/2}a_{k\omega}^{R\dagger} - e^{\pi\omega/2}a_{k\omega}^L\right]\,. \tag{2.84}$$

Intuitively, this is the reflection of the fact that the region W_F can be reached both from W_R by left-moving waves and from W_L by right-moving waves. Note that the right hand sides contain both the creation and the annihilation operators and hence these relations constitute the Bogoliubov transformations between the Rindler modes and the Unruh modes.

As an application of the formulas (2.83) and (2.84), let us express the W_F vacuum $|0\rangle_F$, which is the same as the Minkowski vacuum $|0\rangle_M$, in terms of the states in the W_R and W_L frames. The condition that $|0\rangle_F$ must be annihilated by $a_{k\omega}^F$ and $a_{k,-\omega}^F$ can be expressed in the form

$$a_{k\omega}^L|0\rangle_F = e^{-\pi\omega}a_{k\omega}^{R\dagger}|0\rangle_F\,, \qquad a_{k\omega}^R|0\rangle_F = e^{-\pi\omega}a_{k\omega}^{L\dagger}|0\rangle_F\,. \tag{2.85}$$

The solution is

$$|0\rangle_F = |0\rangle_M = \mathcal{N}\exp\left(\int d^2k\int_0^\infty d\omega e^{-\pi\omega}a_{k\omega}^{L\dagger}a_{k\omega}^{R\dagger}\right)|0\rangle_L \otimes |0\rangle_R\,, \tag{2.86}$$

where $\mathcal{N}$ is a normalization factor[11] and $|0\rangle_{L,R}$ are the vacua for the W_L and W_R frames defined by $a_{k\omega}^L|0\rangle_L = 0, a_{k\omega}^R|0\rangle_R = 0$, for all k and positive ω. They are known as the Rindler vacua.

Let us make a few remarks on the relation between the field expressed in the Minkowski frame and in the combined W_R and W_L frame.

- In the context of the discussions of the entanglement and the entropy thereof, instead of the expression (2.86) for the Minkowski vacuum, a simpler formula of the form

$$|0\rangle_M = \sum_{n=0}^\infty e^{-\omega\pi n}|n\rangle_L \otimes |n\rangle_R \tag{2.87}$$

  is often quoted in the literature. This of course is an expression for the two dimensional toy model with only one frequency kept. The full expression (2.86) for four dimensions can be written in a form similar to the above after discretizing the energy ω and the momenta k and expanding the exponential.

---

[11]The normalization constant $\mathcal{N}$ is divergent as it stands. To make it finite, one must discretize k and ω and regularize the infinite sum.

- By using the relations (2.57), (2.83) and (2.84), one can express $\phi^M(t_M, x^1, x)$ in terms of the modes of W_L and W_R. An important check is if $\phi^M(t_M, x^1, x)$ so constructed depends only on the modes of W_L (W_R) when $x^1 < 0$ ($x^1 > 0$). In Appendix B.3, we shall sketch a proof [12], which turned out to require a careful treatment of the proper analytic continuation.

2.7 Representation of Poincaré algebra for various observers

2.7.1 Poincaré algebra for the $1 + 1$ dimensional subspace

Evidently, the Poincaré symmetry of the flat Minkowski space is a fundamental symmetry governing, above all, the structure of correlation functions. Although the quantum generators of the Poincaré algebra are well-known in the ordinary Minkowski frame, their forms are non-trivial in terms of the modes of the observers in the W_F, W_R and W_L wedges and have not been discussed in the literature. In this subsection, we shall construct them by using the relations among the modes of the various observers established in the previous subsections.

As will be described in the next section, the vicinity of the horizon of the four dimensional Schwarzschild black hole of our interest has the structure of the $1 + 1$ dimensional flat space $\mathbb{R}^{1,1}$. For that reason, in what follows we shall focus exclusively on the generators and the algebra pertaining to such subspace of the four-dimensional Minkowski space. In terms of the coordinates of the aforementioned observers, the metric of the subspace $\mathbb{R}^{1,1}$ is expressed as

$$ds^2 = -(dt_M)^2 + (dx^1)^2 = z_F^2 dt_F^2 - dz_F^2 = -z_R^2 dt_R^2 + dz_R^2 = -z_L^2 dt_L^2 + dz_L^2. \qquad (2.88)$$

As usual, the Poincaré generators can be constructed in terms of the energy-momentum tensor, which for a scalar field takes the form

$$T^\mu_\nu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} = -\partial^\mu \phi \partial_\nu \phi + \frac{1}{2} \delta^\mu_\nu \partial^\rho \phi \partial_\rho \phi. \qquad (2.89)$$

Here μ, ν, ρ refer to general coordinates. Then, the generators of the Poincaré algebra of $\mathbb{R}^{1,1}$ are the energy H and the momentum P_1 in the first direction

$$H \equiv P_0 = \int d^2 x \int dx^1 T^0_0, \qquad P_1 = \int d^2 x \int dx^1 T^0_1, \qquad (2.90)$$

and the boost generator along the first direction M_{01}

$$M_{01} = \int d^2 x \int dx^1 (x^0 T^{01} - x^1 T^{00}). \qquad (2.91)$$

---

[12]In the basic literature such as [33] and [15], this property appears to be put in by hand rather than derived.

The subscripts 0 and 1 here refer of course to the directions in the Minkowski frame and when quantized the normal-ordering prescription for the modes is taken for granted. Then, in terms of the Minkowski modes, these generators are given by

$$H = \int dk^2 \int dp^1 \, E_{kp^1} a_{kp^1}^{M\dagger} a_{kp^1}^M \,, \tag{2.92}$$

$$P_1 = \int dk^2 \int dp^1 p_1 a_{kp^1}^{M\dagger} a_{kp^1}^M \,, \tag{2.93}$$

$$M_{01} = i \int dk^2 \int dp^1 E_{kp^1} \, a_{kp^1}^{M\dagger} \frac{\partial}{\partial p_1} a_{kp^1}^M \,, \tag{2.94}$$

and can be checked to form the $1+1$ dimensional Poincaré algebra

$$[H, M_{01}] = iP_1 \,, \quad [P_1, M_{01}] = iH \,, \quad [H, P_1] = 0 \,. \tag{2.95}$$

2.7.2   Poincaré generators for W_F observer

Recall that the relation between the Minkowski modes $a_{kp^1}^M$ and the (Unruh) modes $a_{k\omega}^F$ for the W_F observer have been worked out in (2.57) (which is reproduced for convenience below)

$$a_{kp^1}^M = -i \int \frac{d\omega}{\sqrt{2\pi E_{k',p^1}}} \left(\frac{E_{kp^1} - p^1}{E_{kp^1} + p^1} \right)^{\frac{i\omega}{2}} a_{k\omega}^F = \frac{-i}{\sqrt{2\pi E_{k',p^1}}} \int d\omega e^{-i\omega u} a_{k\omega}^F \,, \tag{2.96}$$

where $u \equiv \frac{1}{2} \ln \frac{E_{kp^1} + p^1}{E_{kp^1} - p^1}$ is the "rapidity" variable. Substituting this into (2.94), M_{01} can be rewritten in terms of the W_F modes as

$$M_{01}^F = \int d^2 k \int d\omega \, \omega a_{k\omega}^{F\dagger} a_{k\omega}^F \,. \tag{2.97}$$

Note that this is diagonal in ω and hence interpretable as the "momentum" operator. In Appendix C.2, we show explicitly that by the unitary transformation constructed in (2.79), M_{01} and M_{01}^F are transformed into each other.

Next, let us rewrite the Hamiltoian operator in terms of the Unruh modes. Using the rapidity representation, with $E_{kp^1} = |k| \cosh u$, we get

$$H = \int dk^2 \int dp^1 \, E_{k'p^1} a_{kp^1}^{M\dagger} a_{kp^1}^M$$
$$= \int d^2 k |k| \int d\omega d\omega' \int \frac{du}{2\pi} \cosh u e^{i(\omega'-\omega)u} a_{k\omega'}^{F\dagger} a_{k\omega}^F \,. \tag{2.98}$$

Since the integral over u is divergent and behaves as $\sim e^{|u|}$ at large $|u|$, we should define this integral with a suitable regularization. We adopt the definition

$$\int \frac{du}{2\pi} \cosh u e^{i(\omega-\omega')u} \equiv \lim_{\epsilon \to +0} \int \frac{du}{2\pi} e^{-\epsilon u^2} \cosh u e^{i(\omega-\omega')u} \,. \tag{2.99}$$

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Then, expanding $\cosh u$ in powers, we can rewrite this integral as

$$\lim_{\epsilon \to +0} \int \frac{du}{2\pi} e^{-\epsilon u^2} \cosh u e^{i(\omega'-\omega)u}$$

$$= \lim_{\epsilon \to +0} \int \frac{du}{2\pi} e^{-\epsilon u^2} \sum_n \frac{u^{2n}}{(2n)!} e^{-\epsilon u^2} e^{i(\omega'-\omega)u}$$

$$= \lim_{\epsilon \to +0} \int \frac{du}{2\pi} e^{-\epsilon u^2} \sum_n \frac{(-1)^n}{(2n)!} \left(\frac{\partial}{\partial \omega}\right)^{2n} e^{i(\omega'-\omega)u}$$

$$= \lim_{\epsilon \to +0} \int \frac{du}{2\pi} e^{-\epsilon u^2} \cos \left(\frac{\partial}{\partial \omega}\right) e^{i(\omega'-\omega)u} . \qquad (2.100)$$

The remaining gaussian integral produces a δ function $\delta(\omega'-\omega)$ and hence the Hamiltonian can be written as

$$H^F = \int d^2k |k| \int d\omega d\omega' \cos \left(\frac{\partial}{\partial \omega}\right) \delta(\omega' - \omega) a^{F\dagger}_{k\omega'} a^F_{k\omega}$$

$$= \int d^2k |k| \int d\omega \, a^{F\dagger}_{k\omega} \cos \left(\frac{d}{d\omega}\right) a^F_{k\omega} . \qquad (2.101)$$

In an entirely similar manner, P_1 operator is expressed as

$$P_1^F = -i \int d^2k |k| \int d\omega \, a^{F\dagger}_{k\omega} \sin \left(\frac{d}{d\omega}\right) a^F_{k\omega} . \qquad (2.102)$$

These operators are understood to be used within a matrix element such that the object is infinitely differentiable with respec to ω.

In Appendix C.1, we demonstrate that these operators M_{01}^F, H^F and P_1^F do satisfy the $1+1$ dimensional Poincaré algebra (2.95).

2.7.3   Poincaré generators for W_R and W_L observers

Having derived the expression of the generators in terms of W_F oscillators, we can now write them in terms of the W_R and W_L mode operators using the relations (2.83) and (2.84), that is,

$$a^F_{k\omega} = -\frac{i}{\sqrt{2\sinh \pi \omega}} \left[e^{\pi \omega/2} a^R_{k\omega} - e^{-\pi \omega/2} a^{L\dagger}_{k\omega}\right] , \qquad (2.103)$$

$$a^F_{-k,-\omega} = \frac{i}{\sqrt{2\sinh \pi \omega}} \left[e^{-\pi \omega/2} a^{R\dagger}_{k\omega} - e^{\pi \omega/2} a^L_{k\omega}\right] . \qquad (2.104)$$

The result cleanly separates into the W_R part and the W_L part for each generator and we get

$$M_{01} = M_{01}^R + M_{01}^L , \qquad H = H^R + H^L , \qquad P_1 = P_1^R + P_1^L , \qquad (2.105)$$

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where

$$M_{01}^R = \int d^2k \int_0^\infty d\omega \, \omega a_{k\omega}^{R\dagger} a_{k\omega}^R \,, \qquad\qquad M_{01}^L = -M_{01}^R\big|_{a_{k\omega}^R \to a_{k\omega}^L} \,, \qquad (2.106)$$

$$H^R = \int d^2k \int_0^\infty d\omega \, |k| a_{k\omega}^{R\dagger} \cos\left(\frac{d}{d\omega}\right) a_{k\omega}^R \,, \qquad H^L = H^R\big|_{a_{k\omega}^R \to a_{k\omega}^L} \,, \qquad (2.107)$$

$$P_1^R = -i \int d^2k \int_0^\infty d\omega \, |k| a_{k\omega}^{R\dagger} \sin\left(\frac{d}{d\omega}\right) a_{k\omega}^R \,, \qquad P_1^L = -P_1^R\big|_{a_{k\omega}^R \to a_{k\omega}^L} \,. \qquad (2.108)$$

Because of the relative minus signs for $M_{01}^{R,L}$ and $P_1^{R,L}$, commutators of these generators give precisely the *same* Poincaré algebra in the W_R wedge and in the W_L wedge. Explicitly

$$[H^R, M_{01}^R] = iP_1^R, \quad [P_1^R, M_{01}^R] = iH^R, \quad [H^R, P_1^R] = 0$$
$$[H^L, M_{01}^L] = iP_1^L, \quad [P_1^L, M_{01}^L] = iH^L, \quad [H^L, P_1^L] = 0. \qquad (2.109)$$

Two remarks are in order.

(i) First, M_{01}^R is diagonal in ω, which in this case has the meaning of the energy conjugate to the Rindler time t_R. This clearly shows that the boost generator M_{01}^R is the Hamiltonian for the W_R observer.

(ii) Second, the relative minus sign between M_{01}^R and M_{01}^L simply means that the "time" flows in opposite directions in W_R and W_L.

These remarks are expressed by the following simple relations:

$$e^{i\xi M_{01}^R} a_{k\omega}^R e^{-i\xi M_{01}^R} = e^{-i\omega\xi} a_{k\omega}^R \,, \qquad (2.110)$$

$$e^{i\xi M_{01}^L} a_{k\omega}^L e^{-i\xi M_{01}^L} = e^{i\omega\xi} a_{k\omega}^L \,. \qquad (2.111)$$

Thus, acting on the field, the boost generator indeed induces the Rindler time evolution in each wedge as shown below:

$$e^{i\xi M_{01}^R} \phi^R(t_R, z_R, x) e^{-i\xi M_{01}^R} = \int_0^\infty d\omega \int d^2k N_\omega^R [K_{i\omega}(|k|z_R) e^{ikx} e^{-i\omega(t_R+\xi)} a_{k\omega}^R + \text{h.c.}] \,,$$

$$e^{i\xi M_{01}^L} \phi^L(t_L, z_L, x) e^{-i\xi M_{01}^L} = \int_0^\infty d\omega \int d^2k N_\omega^L [K_{i\omega}(|k|z_R) e^{-ikx} e^{i\omega(t_L+\xi)} a_{k\omega}^L + \text{h.c.}] \,.$$

3 Quantization in an eternal Schwarzschild black hole by various observers

As was emphasized in the introduction, the main aim of this paper is to study the structure of the Hilbert spaces of the scalar field near the horizon of the Schwarzschild black hole quantized in the frames of different observers. This is made possible largely due to the fact that such near-horizon geometry has the structure of the flat Minkowski space, to be recalled shortly. This allows us to make use of the knowledge of the quantization in various frames which has been reviewed, with some additional new information, in the previous section.

Now in studying the quantization around the horizon of a black hole, it will be important to distinguish the two cases, namely the case of the eternal (*i.e.* two-sided) black hole and the more physical one where the (one-sided) black hole is produced by a collapse of matter (or radiation). There are essential differerences between the two.

In this section, we analyze the simpler case of the eternal Schwarzschild black hole.

3.1 Flat geometry around the event horizon of a Schwarzschild black hole

Let us first recall how the flat geometry emerges in the vicinity of the event horizon of a Schwarzschild black hole.

We denote the metric for the four-dimensional Schwarzschild of mass M in the asymptotic coordinate in the usual way:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)d\tilde{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\Omega^2\,. \tag{3.1}$$

The notations are standard, except that we set the Newton constant G to one and used "$\tilde{t}$" to denote the Schwarzschild time. This will be later rescaled to "t" to denote the Minkowski time.

First we consider the region $\mathrm{W_R}$ outside the horizon. It is convenient to introduce a positive coordinate "z" which measures the proper radial distance from the horizon:

$$z \equiv \int_{2M}^{r} \sqrt{g_{rr}(r')}dr' = \int_{2M}^{r} \frac{1}{\sqrt{1 - \frac{2M}{r'}}}dr'\,. \tag{3.2}$$

Near the horizon at $r = 2M$, we write r as $r = 2M + y$, expand z in powers of y in the form $z = ay^{1/2}(1 + by + \cdots)$ and then solve for y in terms of z. After a simple calculation we obtain

$$r = 2M + \frac{M}{8}\left(\frac{z}{M}\right)^2 - \frac{M}{384}\left(\frac{z}{M}\right)^4 + \mathcal{O}((z/M)^6)\,. \tag{3.3}$$

Now if keep up to the second term of this expansion, the Schwarzschild the metric becomes

$$ds^2 \simeq -z^2(dt)^2 + dz^2 + r^2(z)d\Omega^2 \tag{3.4}$$

$$t \equiv \frac{\tilde{t}}{4M}, \qquad d\Omega^2 = d\theta^2 + \sin\theta^2 d\phi^2. \tag{3.5}$$

Further, focusing on the small two dimensonal region perpendicular to the radial direction around $\theta = 0$, we can parametrize it by the coordinates

$$x^2 = 2M\theta\cos\phi, \qquad x^3 = 2M\theta\sin\phi. \tag{3.6}$$

Then in this region the metric further simplifies and becomes identical to the Rindler metric for W_R given in (2.5)

$$ds^2 = -z^2 dt^2 + dz^2 + (dx^2)^2 + (dx^3)^2, \tag{3.7}$$

which expresses (a portion of) the flat spacetime[13].

To see the region of validity of this approximation, let us find out the condition under which we can neglect the third term of the expansion (3.3) compared to the second. A simple calculation shows that the condition is

$$\frac{z}{M} \ll 4\sqrt{3} \sim 7, \tag{3.8}$$

showing that the flat space approximation is good for z up to the order of the Schwarzschild radius $\mathcal{O}(M)$ out from the horizon.

For the other regions W_L, W_F, and W_P, by appropriate analytic continutations, we obtain similar flat space form of the metric of appropriate signature, as already displayed in Sec. 2. In particular, we should remember that as we go from W_R to W_F the role of time and space variables are interchanged.

3.2 Exact treatment for the transverse spherical space

The approximation of the vicinity of the horizon as a four-dimensional flat Minkowski space is certainly a great advantage, as long as we are interested only in the quantities determined by the local properties of the fields. However, as we have repeatedly emphasized, in quantum treatment the concept of states created by the mode operators is a global one and that is precisely what we are interested in. It turns out that the inadequacy of the flat approximation is particularly troublesome for the two-dimensional transverse space, since

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[13]The approximation of taking r to be the fixed value $2M$ in (3.6) is admissible, since in the expression $(dx^2)^2 + (dx^3)^2$ the radial coordinate appears in the forms $r^2d\theta^2, r^2d\phi^2, dr^2$ and $rdrd\theta$. For these expressions, the order $\mathcal{O}(z^2/M^2)$ terms are safely neglected.

the orthogonality relation needed to extract the modes from the fields requires integration over the entire range of (x^2, x^3) expressing the flat 2-space, which is unjustified for large values of these coordinates.

The obvious cure for this part of the problem is to replace the expansion in terms of the plane waves by the spherical harmonics $Y_{lm}(\theta, \varphi)$. Thus, instead of $M^{1,3}$, we will be dealing with the spacetime $\mathbb{R}^{1,1} \times \mathbb{S}^2_{2M}$, where the subscript $2M$ denotes the radius of the sphere.

Explicitly, we can write the general expansions of a massless scalar and its conjugate in the vicinity of the horizon in the form

$$\phi(t, x^1, \Omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_{-\infty}^{\infty} \frac{dp^1}{\sqrt{4\pi E_{k_l p^1}}} e^{ip^1 x^1 - iE_{k_l p^1} t} Y_{lm}(\Omega) a_{lmp^1} + \text{h.c.}, \tag{3.9}$$

$$\pi(t, y^1, \Omega') = -i \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} \int_{-\infty}^{\infty} \frac{dq^1}{\sqrt{4\pi E_{k_{l'} q^1}}} E_{k_{l'} q^1} e^{iq^1 y^1 - iE_{k_{l'} q^1} t} Y_{l'm'}(\Omega') a_{l'm'q^1} + \text{h.c.} \tag{3.10}$$

where $\Omega = (\theta, \varphi)$. The energy $E_{k_l p^1}$ is determined in terms of p^1 and l by the equation of motion as

$$E^2_{k_l p^1} = (p^1)^2 + k_l^2, \qquad k_l \equiv \frac{\sqrt{l(l+1)}}{2M}. \tag{3.11}$$

The equal time canonical commutation relation takes the form

$$\left[\pi(t, x^1, \Omega), \phi(t, y^1, \Omega')\right] = -i\delta(x^1 - y^1)\delta(\cos\theta - \cos\theta')\delta(\varphi - \varphi'). \tag{3.12}$$

The orthogonality for $Y_{lm}(\Omega)$ is

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta Y_{lm}^*(\theta, \varphi) Y_{l'm'}(\theta, \varphi) = \delta_{ll'}\delta_{mm'}, \tag{3.13}$$

while the completeness reads

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}^*(\theta, \varphi) Y_{lm}(\theta', \varphi') = \delta(\cos\theta - \cos\theta')\delta(\varphi - \varphi'). \tag{3.14}$$

Using the orthogonality relation, we can extract the modes as

$$a_{lmp^1} = \int \frac{dx^1}{\sqrt{4\pi E_{k_l p^1}}} \int d\Omega Y_{lm}^*(\Omega) e^{-ip^1 x^1 + iE_{k_l p^1} t} i\overleftrightarrow{\partial_t} \phi(t, x^1, \Omega), \tag{3.15}$$

$$a_{lmp^1}^\dagger = \int \frac{dx^1}{\sqrt{4\pi E_{k_l p^1}}} \int d\Omega Y_{lm}(\Omega) e^{ip^1 x^1 - iE_{k_l p^1} t} \frac{1}{i}\overleftrightarrow{\partial_t} \phi(t, x^1, \Omega). \tag{3.16}$$

From the canonical commutation relation (3.12), the modes satisfy

$$\left[a_{lmp^1}, a_{l'm'q^1}^\dagger\right] = \delta_{ll'}\delta_{mm'}\delta(p^1 - q^1), \qquad \text{rest} = 0. \tag{3.17}$$

In summary, the expansion in flat space described in Sec. 2 can be converted to the present case by the simple replacements

$$\int \frac{d^2k}{(2\pi)^2} e^{ikx}(**)a_{kp^1} \quad \longrightarrow \quad \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\Omega)(**)a_{lmp^1} \,, \tag{3.18}$$

$$E_{kp^1} = \sqrt{k^2 + (p^1)^2} \quad \longrightarrow \quad E_{k_l p^1} = \sqrt{k_l^2 + (p^1)^2} \,, \quad k_l^2 \equiv \frac{l(l+1)}{(2M)^2} \,, \tag{3.19}$$

$$a_{kp^1} \quad \longrightarrow \quad a_{lmp^1} \,. \tag{3.20}$$

As the behavior of the scalar field on the transverse spherical surface near the horizon is treated exactly as above, we need only be concerned with the dependence on the remaining two dimensions (t_M, x^1). Thus from now on, we will use expressions such as "flat approximation" or "flat space" to refer only to the two dimensional part near the horizon within $\mathbb{R}^{1,1}$.

3.3 Quantization in the frame of freely falling observer near the horizon

Among many interesting questions which stem from the observer dependence of the quantization around a black hole, perhaps the most provocative one is whether the freely falling observer, hereafter abbreviated as FFO, sees a different Hilbert space structure for the quantized scalar field before and after he/she passes through the horizon: In other words, whether the equivalence principle for the field is affected by the quantum effects or not.

In this subsection, we will perform some preparatory computations in the frame of FFO, who crosses the horizon along various directions in the Penrose diagram, *i.e.* with various velocities.

First, let us briefly describe how the geodesic of a massive classical particle (which represents a FFO) near the horizon maps to the motion in the flat coordinate system obtained by the non-linear transformation of the previous subsection. Although the final answer should be a straightline in the flat coordinate system, as the geodesic should map to a geodesic, it is instructive to see the physical meaning of this mapping.

Consider first the motion in W_R. The geodesic equation in the radial direction of a massive particle (with mass set to unity) in the Schwarzschild spacetime in the region W_R takes the form

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left(1 - \frac{2M}{r}\right) = \frac{1}{2}E^2 \,, \tag{3.21}$$

where E is a constant of motion given by

$$E = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} \,. \tag{3.22}$$

Here t is the asymptotic time and τ is the proper time. Restricting to the region near the horizon, we can approximate r by $r \simeq 2M + (z_R^2/8M)$ as worked out in (3.3). Then, from (3.22) one can express $d\tau$ in terms of t and z_R and rescaling t like $t = 4Mt_R$ as in (3.5), we can easily rewrite the geodesic equation as

$$\left(\frac{1}{z_R} \frac{dz_R}{dt_R} \right)^2 + b^2 z_R^2 = 1 \,, \qquad b \equiv \frac{1}{4ME} \,. \tag{3.23}$$

This differential equation for z_R as a function of t_R is easily solved to give[14]

$$z_R = \frac{2c}{c^2 e^{t_R} + b^2 e^{-t_R}} \,, \qquad c > 0 \,, \tag{3.24}$$

where c is a positive integration constant. This shows that z_R vanishes as $t_R \to \pm\infty$, meaning that the trajectory starts and ends at the horizon. The physical picture is that, due to the gravitational attraction of the black hole, the trajectory which starts out at the horizon at $t_R = -\infty$ with some initial velocity goes out to a certain maximum distance (actually $z_R = 4ME$) away from the horizon where it stops and then gets pulled back to the horizon at $t_R = \infty$.

Now let us rewrite this motion (3.24) in terms of the flat Minkowski coordinates (t_M, x^1) related to (z_R, t_R) by $t_M = z_R \sinh t_R$, $x^1 = z_R \cosh t_R$ as in (2.3). Then, we get

$$t_M = -\frac{1}{\beta} x^1 + X \,, \tag{3.25}$$

$$\beta \equiv \frac{c^2 - b^2}{c^2 + b^2} \,, \qquad X \equiv \frac{2c}{c^2 - b^2} \,. \tag{3.26}$$

As expected, this describes a family of timelike straight line trajectories, with the velocity β. To construct an orthogonal coordinate system with the trajectories above as specifying the time direction, we must supply spacelike lines perpendicular (in the Lorentz sense) to them. Clearly they are of the form

$$t_M = -\beta x^1 + T \,, \tag{3.27}$$

where T is a parameter. Thus by changing the values of X and T we span (a part of) the Minkowski space. In other words, (T, X) serve as new orthogonal coordinates. In fact better coordinates are the rescaled ones (t_β, x_β^1) defined in the following way:

$$t_\beta \equiv \gamma T \,, \qquad x_\beta^1 \equiv \gamma \beta X \,, \qquad \gamma \equiv \frac{1}{\sqrt{1 - \beta^2}} \,, \tag{3.28}$$

$$ds^2 = -dt_\beta^2 + (dx_\beta^1)^2 \,. \tag{3.29}$$

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[14]Actually, there is another solution with the sign in front of t_R's flipped. But since they are related simply by changing the sign of t_R, we deal with the one displayed below without loss of generality.
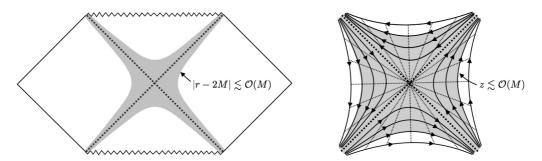
Then the relation to the canonical Minkowski variables (t_M, x^1) are obtained from (3.25) and (3.27) and can be written as

$$\begin{pmatrix} t_\beta \\ x_\beta^1 \end{pmatrix} = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} t_M \\ x^1 \end{pmatrix} . \tag{3.30}$$

This is nothing but the Lorentz boost by the velocity β in the negative x^1 direction.

One can perform a similar analysis of the geodesic in W_F region sharing the horizon as the boundary with W_R. The outcome of the study is that the geodesics which hit the same point on this common boundary from inside and the outside with the same velocity β are actually one and the same straightline which is obtained by the Lorentz boost of the trajectory along the time axis in the canonical Minkowski coordinate. Physically this must be the case since the FFO must be able to go through the horizon freely due to the *classical* equivalence principle.

With this preparation, let us now discuss the quantization and the mode expansion of the free scalar field by an FFO in the vicinity of the horizon where the flat space approximation for the dependence on (t_M, x^1) is valid. In the case of the two-sided eternal Schwarzschild black hole studied in this section, we have both W_R and W_L regions and approximately flat region near the horizons can be dipicted as the shaded region in Figure 3.1.

Two-sided Schwarzschild black hole   Rindler coordinate (flat spacetime)

Figure 3.1: Approximately flat regions, shown in gray, near the horizon of the two-sided Schwarzschild black hole and the corresponding region in the Rindler coordinate of the flat spacetime.

In this region the general solution for a scalar field as seen by an FFO is

$$\phi^M(t_M, x^1, \Omega) = \sum_{l,m} \int_{-\infty}^{\infty} \frac{dp^1}{\sqrt{2E_{klp^1}}} \left(e^{ip^1 x^1 - iE_{klp^1} t_M} Y_{lm}(\Omega) a_{lmp^1}^M + \text{h.c.} \right) . \tag{3.31}$$

This expression is perfectly valid *locally* but as we try to extract the mode operators $a_{lmp^1}^M$ and their conjugate and check that they obey the usual commutation relations, we

encounter a problem of the same nature as occured for the use of the flat coordinates (x^2, x^3). Namely, such an extraction requires orthogonality relations for the plane waves, which involves integration over an infinite range for the spatial variable x^1. As such a range is not within the flat space region, it appears to be quite difficult to solve this problem.

The observation which allows us to overcome this obstacle is that regions of infinite range do exist around the horizon along the lightcones in the (t_M, x^1) space. Technically, however, the quantization using the exact lightcone variable as the time is rather singular. Therefore, we shall make a very large (but not infinite) two-dimensional Lorentz boost so that the x^1-axis is rotated to the direction which is almost lightlike yet still slightly spacelike. Then assuming the usual regularization that the scalar field vanishes at $t_M = \pm\infty$, we can integrate along this new x^1 axis, which is practically contained in the flat region, and extract the modes. Since what we used here is a Lorentz transformation, the exponent is invariant, while the modes are transformed in a well-known simple way, namely

$$a'^M_{lmp'^1} \sqrt{E'_{k_l p'^1}} = a^M_{lmp^1} \sqrt{E_{k_l p^1}}, \qquad \text{(same for the conjugates)}, \qquad (3.32)$$

where prime signifies the Lorentz transformed quantities. The mode operators satisfy the usual commutation relations, *i.e.* $[a'_{lmp'^1}, a'^{\dagger}_{l'm'q'^1}] = \delta_{ll'}\delta_{mm'}\delta(p'^1 - q'^1)$ etc. This immediately tells us that the number of degrees of freedom observed by the FFO in the horizon region is exactly the same as that of the scalar field in the usual Minkowski space and the structure of the Hilbert space is unchanged across the horizon. In this sense, *the equivalence principle is still valid quantum mechanically* around the eternal black hole.

3.4 Relation between the quantization by a freely falling observer and the stationary observers in $\mathbf{W_F}$ and $\mathbf{W_P}$

As the argument for $\mathrm{W_P}$ is the same as that for $\mathrm{W_F}$ we will concentrate on the case of $\mathrm{W_F}$ observer.

In the approximately flat region near the horizon, the scalar field ϕ^F can be expanded in modes simply like

$$\phi^F(t_F, z_F, \Omega) = \sum_{l,m} \int_{-\infty}^{\infty} d\omega N_\omega^F \left(e^{-i\omega t_F} H_{i\omega}^{(2)}(k_l z_F) Y_{lm}(\Omega) a_{lm\omega}^F + \text{h.c.} \right). \qquad (3.33)$$

Contrary to the case of the Minkowski frame discussed above, the extraction of the modes in $\mathrm{W_F}$ is straightforward. This is because the equal-time spacelike lines near the horizon are entirely contained in the approximately flat region, as is clear from the Figure 3.1.

Therefore orthogonality relations for the Hankel functions can be used just as in the case of the entire Minkowski space, described in Sec. 2.5.

This means that in the flat region around the horizon, the number of modes are the same between W_F observer and the FFO. More explicitly, the relations between the mode operators are just as in the case of the Minkowski space (with k replaced by lm taken for granted). This is particularly clear in the rapidity representation given in (2.55) and (2.56). Since $|k|\cosh u$ in the definition (2.51) is the energy E, the operator a_{ku}^M is Lorentz invariant as seen from (3.32), which means $a'^{M}_{ku'} = a_{ku}^M$, where $u' = u + \xi$, where ξ is the rapidity for the boost. On the other hand, the invariance of ϕ^F and z_F and $t_F \to t_F + \xi$ under the Lorentz transformation in (2.39) dictates that we should have $a'^{F}_{k\omega} = e^{i\omega\xi}a_{k\omega}^F$. With the angular-momentum indices explicitly implemented, we have, under the Lorentz transformation,

$$a'^{M}_{lm,u+\xi} = a_{lmu}^M\,, \qquad a'^{F}_{lm\omega} = e^{i\omega\xi}a_{lm\omega}^F\,. \tag{3.34}$$

It is easy to see that this is indeed compatible with the Fourier transform relation (2.55) with k replaced by lm.

3.5 Relation between the quantization by a freely falling observer and the stationalry observers in W_R and W_L

We now come to the more difficult situation of the quantization from the viewpoints of the W_R (and W_L) observer in the flat region. Expansion of the general solution into modes using the $K_{i\omega}$ functions is the same as in the Minkowski space and the canonical quantization condition for the fields can be imposed. But the extraction of the mode operators $a_{lm\omega}^M, a_{lm\omega}^{M\dagger}$ and verifying that they satisfy the canonical commutation relations cannot be performed explicitly. In contrast to the case of W_F discussed in the previous subsection, there is no spacelike line covering the interval $t_R \in [-\infty, \infty]$ within the flat region, and hence we cannot use the orthogonality relation to express the mode operators in terms of the fields.

What we can check easily is that, if we assume the canonical form of the commutation relations for the modes as in the flat space, then by using the completeness relation, *which is a local relation*, the correct canonical commutation relations for the fields are reproduced. This shows the self-consistency of the assumption.

Actually, we can argue that the relation between a^R and a^M should be the same as in the full Minkowski space in the following two ways:

(i) In the flat region, using completeness, we can reexpand the field ϕ^R, which contains a^R and $a^{R\dagger}$ in terms of the plane waves, *i.e.* in terms of the modes of ϕ^M. In this calculation,

we only need to use integration over the momenta. Now, as described in Sec. 3, we can use the orthogonality of the plane waves along the contour which by a suitable Lorentz transformation is brought within the flat region extending to infinity near the horizon and extract a^M modes. Along such a line, we can relate a^M's with a^R's as in the full Minkowski space. Then Lorentz transforming back this relation, we should be able to express a^R's in terms of a^M's in any flat region around the horizon.

(ii) Another argument goes as follows. For simplicity, consider the case where we try to use the orthogonality integral along the spacelike straight line at $t_M = 0$ extending from $x^1 = -\infty$ to $x^1 = \infty$. This passes both W_L and W_R and only a portion of the contour is within the flat region. Outside the flat region, the eigenfunctions $f_{k_l\omega}(z_{R,L})$ satisfying the equation of motion starts to differ *continuously* from the modified Bessel functions $K_{i\omega}(k_l z_{R,L})$. But since the differential equation expressing the equation of motion does not acquire any new singularities, one expects that such deformed eigenfunctions continue to satisfy appropriate form of orthogonality relations. Then, using them, one can extract $a^{R,L}$'s from the fields and compute the commutation relations among them. These relations should reduce (continuously) to the usual commutation relations in the flat region, as they must lead, with the use of the completeness relation, to the correct canonical commutation relations for the fields expandable in terms of the modified Bessel functions in such region.

These arguments indicate that as far as the flat region near the horizon is concerned the relations between the modes for the FFO and the observers in various Rindler frames should be the same as those already exhibited in Sec. 2 for the fully flat case, with the replacement of the linear momentum label k by the angular momentum label lm.

4 Quantization in a Vaidya model of a physical black hole by various observers

Black holes of more physical interest are the ones formed by a collapse of matter as actually occurs in nature. They are "one-sided" and have rather different spacetime structures compared with the two-sided eternal black holes discussed in the previous section.

In this section, we investigate how the observers in various frames quantize a massless scalar field in the simplest model of Schwarzschild black hole of such a type, namely the so-called Vaidya spacetime [28–30][15], created by the collapse of a thin spherical shell of matter at the speed of light, often referred to as a shock wave.

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[15] For a review, see for example [31].

4.1 Vaidya model of a physical black hole and the effect of the shock wave on the field as a boundary condition

4.1.1 Vaidya model of a physical black hole

Let us begin by recalling the basics of such a Vaidya spacetime. After a black hole is formed by the spherical collapse, by the Birkohoff's theorem, the metric outside the horizon is always that of the Schwarzschild black hole. On the other hand, for the simplest situation above, the metric inside is isomorphic to a part of the flat Minkowski space. Thus the Penrose diagram of the entire spacetime is obtained by gluing these two types of geometries along the light-like line representing the falling shell, as shown in Figure 4.1.
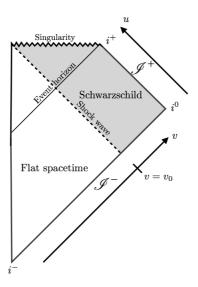
Figure 4.1: Penrose diagram of the simplest Vaidya spacetime. It consists of two parts. One is the flat region inside the matter shell ($v < v_0$) shown in white. The other is the Schwarzschild spacetime outside the matter shell ($v > v_0$) shown in gray.

The Vaidya metric is a solution of the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}\,, \tag{4.1}$$

with the energy momentum tensor

$$T_{vv} = \frac{M}{4\pi r^2}\delta(v - v_0). \tag{4.2}$$

The delta function at $v = v_0$ represents a shock wave induced by the matter collapsing along the lightlike direction u. The metric of the Vaidya spacetime is described by

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right)dv^2 + 2drdv + r^2d\Omega^2\,, \tag{4.3}$$

where

$$
\begin{cases}
m(v) = 0 & \text{for } v < v_0 \\
m(v) = M & \text{for } v > v_0
\end{cases}
. \tag{4.4}
$$

The metric above consists of two parts, one of which corresponds to the region inside the shock wave $v < v_0$ and the other describes the outside, *i.e.* the region $v > v_0$. The solution inside the shock wave is actually a flat spacetime described by

$$
ds^2 = -dv^2 + 2drdv + r^2d\Omega^2 = -dt^2 + dr^2 + r^2d\Omega^2 , \tag{4.5}
$$

where $v = t + r$ is a lightcone coordinate. This is expected from the spherical symmetry of the matter shell. The solution for the region $v > v_0$ is the Schwarzschild black hole in the Eddington-Finkelstein coordinate

$$
ds^2 = -\left(1 - \frac{2M}{r}\right)dv^2 + 2drdv + r^2d\Omega^2 , \tag{4.6}
$$

as dictated by the Birkohoff's theorem. This metric can be transformed into the Schwarzschild form

$$
ds^2 = -\left(1 - \frac{2M}{r}\right)d\tilde{t}^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2d\Omega^2 \tag{4.7}
$$

by the coordinate transformation $v = \tilde{t} + r^*$. Here $\tilde{t}$ is the Schwarzschild time and r^* is the tortoise coordinate which is defined by

$$
r^* = \int \frac{dr}{1 - \frac{2M}{r}} = r + 2M \ln\left(\frac{r}{2M} - 1\right) . \tag{4.8}
$$

Notice that the two time coordinates, t inside the shell and $\tilde{t}$ outside, are different. They are related as

$$
t + r = \tilde{t} + r^* \qquad \text{at } v = v_0 ,
$$
$$
\Leftrightarrow \quad \tilde{t} = t - 2M \ln\left(\frac{v_0 - t}{2M} - 1\right) . \tag{4.9}
$$

In the subsequent sections, we mainly focus on the special limit of the Vaidya spacetime for which v_0 is vanishingly small for the technical reason that the analysis is much easier than the general case of finite v_0 and can be explicitly performed.

4.1.2 Effect of the shock wave on the scalar field as a regularized boundary condition

In order to be able to study the quantization of a scalar field in an explicit manner, in what follows we shall (i) make a reasonable assumption about the effect of the matter shock

wave on the field and (ii) implement it in a well-defined way by making a regularization which replaces the lightlike trajectory by a slightly timelike one.

As for (i), since we focus on the Schwarzschild region outside of the locus of the shock wave, the effect of the shock wave on the scalar field $\phi(x)$ should be taken into account by an imposition of an effective boundary condition on $\phi(x)$ along the trajectory of the shock wave, which must be consistent with the bulk equation of motion. Such boundary conditions are either Dirichlet or Neumann. This depends on the nature of the interaction between the shock wave and the scalar field and for definiteness in this work we adopt the Dirichlet condition and demand that $\phi(x)$ vanishes[16] along the boundary.

Next, let us elaborate on the point (ii). If we take the boundary to be strictly lightlike, *i.e.* along $t_M = -x^1$, there is a complication for the spherical mode with zero angular momentum, for which $k_l = 0$. Thus this component of the scalar field becomes massless in two-dimensions and the future directed massless field satisfying the Dirichlet condition along the light-like line above can only be right-moving and hence chiral. As is well-known, quantization of a chiral scalar in two-dimensions is notoriously troublesome and we would like to avoid it. A physically natural regularization is to endow an infinitesimal mass to the falling matter so that the trajectory is slightly timelike. Then the boundary condition can be treated in a non-singular manner by the standard canonical quantization procedure.

Another advantage for making such a regularization is the following. As it will become evident, the effect of the boundary condition on the quantization can be easily taken into account in the frame of FFO moving in the direction of the shock wave. When this direction is slightly timelike, we can change it by a Lorentz transformation into the case for a general FFO moving with any velocity. On the other hand, even if we could manage to treat the case of the strictly lightlike shock wave and the FFO moving along such a direction, we cannot relate such an observer by a Lorentz boost to a general FFO moving with a finite velocity.

4.2 Quantization of the scalar field with a boundary condition by a freely falling observer

In this section, we explicitly perform the quantization of a scalar field with the boundary condition imposed along a slightly timelike line from the point of view of FFO's traversing the horizon with various velocities.

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[16]For a massless scalar, by using the invariance of the action under a constant shift, we can do so without loss of generality.

4.2.1 Three useful coordinate frames and the imposition of a boundary condition

In what follows, we will concentrate on the flat two dimensional portion in $\mathbb{R}^{1,1}$ and introduce three flat coordinates related by Lorentz transformations. One is the canonical coordinates (t, x^1), (where we use t for t_M for simplicity in this subsection) for which t and x^1 axes respectively run vertically and horizontally. The second is the coordinates $(\hat{t}, \hat{x}^1)$, where the $\hat{t}$ axis runs almost lightlike but slightly timelike direction. To go from (t, x^1) to $(\hat{t}, \hat{x}^1)$, we make a large Lorentz transformation of the form

$$\begin{pmatrix} \hat{t} \\ \hat{x}^1 \end{pmatrix} = \Lambda_\epsilon \begin{pmatrix} t \\ x^1 \end{pmatrix}, \qquad \Lambda_\epsilon = \hat{\gamma} \begin{pmatrix} 1 & \hat{\beta} \\ \hat{\beta} & 1 \end{pmatrix}, \tag{4.10}$$

$$\hat{\beta} = 1 - \epsilon, \qquad \hat{\gamma} = \frac{1}{\sqrt{1 - \hat{\beta}^2}} \simeq \frac{1}{\sqrt{2\epsilon}}, \tag{4.11}$$

where $\epsilon(> 0)$ is an infinitesimal parameter. Thus, the explicit transformations are

$$\hat{t} = \frac{1}{\sqrt{2\epsilon}}((1 - \epsilon)x^1 + t), \qquad \hat{x}^1 = \frac{1}{\sqrt{2\epsilon}}((1 - \epsilon)t + x^1). \tag{4.12}$$

We shall take the boundary line to be the one expressed by (see Figure 4.2)

$$\hat{x}^1 = 0 \quad \Leftrightarrow \quad t = -\frac{1}{1 - \epsilon}x^1. \tag{4.13}$$

and demand that $\phi(\hat{x}^1 = 0) = 0$.

Figure 4.2: Slightly timelike boundary line (shown in red).
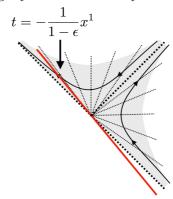
We should remark that this corresponds to the case where the shell of matter collapses along the line for which the so-called tortoise light-cone coordinate $v^* = t + r^*$, where $r^* \equiv \int dr/(1 - (2M/r))$, takes a very large negative constant value compared to the scale of the Schwarzschild radius $2M$. An example is the case where $t \to -\infty$. The reason

for this rather special choice is strictly for technical convenienece: Such a trajectory is contained entirely within the region where the flat space approximation is valid and hence the computations can be done explicitly and reliably. We can deal with a general FFO later by making a Loretz transformation, as explained below. As far as the qualitative conclusions are concerned, a constant shift in v^* should not affect the quantum property of the scalar field drastically, because the Dirichlet condition $\phi = 0$ along the matter trajectory, as we shall see, will act just like a reflecting wall for the scalar field.

The third set of coordinates to be introduced is $(\tilde{t}, \tilde{x}^1)$, where $\tilde{t}$ is the axis along which a FFO travels with a general velocity $\tilde{\beta}$, which can be positive or negative. He/she quantizes the scalar field with $\tilde{t}$ as the time. This frame is defined to be related to the canonical frame by a Lorentz transformation

$$
\begin{pmatrix} \tilde{t} \\ \tilde{x}^1 \end{pmatrix} = \tilde{\Lambda} \begin{pmatrix} t \\ x^1 \end{pmatrix} = \tilde{\gamma} \begin{pmatrix} 1 & \tilde{\beta} \\ \tilde{\beta} & 1 \end{pmatrix} \begin{pmatrix} t \\ x^1 \end{pmatrix},
\tag{4.14}
$$

$$
\tilde{t} = \tilde{\gamma}(t + \tilde{\beta}x^1), \qquad \tilde{x}^1 = \tilde{\gamma}(x^1 + \tilde{\beta}t).
\tag{4.15}
$$

It will also be convenient to relate the frame $(\tilde{t}, \tilde{x}^1)$ with $(\hat{t}, \hat{x}^1)$ directly. We shall write this relation as

$$
\begin{pmatrix} \hat{t} \\ \hat{x}^1 \end{pmatrix} = \Lambda \begin{pmatrix} \tilde{t} \\ \tilde{x}^1 \end{pmatrix}, \qquad \Lambda = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix},
\tag{4.16}
$$

$$
\hat{t} = \gamma(\tilde{t} + \beta\tilde{x}^1), \qquad \hat{x}^1 = \gamma(\tilde{x}^1 + \beta\tilde{t}),
\tag{4.17}
$$

where Λ, in terms of the Lorentz transformations already introduced in (4.10) and (4.14), is the combination $\Lambda = \Lambda_\epsilon \tilde{\Lambda}^{-1}$. For infinitesimal ϵ, the relations between (β, γ) and $(\tilde{\beta}, \tilde{\gamma})$ can be approximated as

$$
\beta \simeq 1 - \frac{1 + \tilde{\beta}}{1 - \tilde{\beta}}\epsilon, \qquad \gamma \simeq \frac{\tilde{\gamma}(1 - \tilde{\beta})}{\sqrt{2\epsilon}}.
\tag{4.18}
$$

4.2.2 Quantization of the scalar field satisfying the boundary condition by a FFO in the $(\hat{t}, \hat{x}^1)$ frame

We begin with the quantization in the $(\hat{t}, \hat{x}^1)$ frame. Since the boundary condition is imposed along the line $\hat{x}^1 = 0$, obviously the quantization is easiest in such a frame. More importantly, (regularizing the scalar field to vanish at infinity, as usual) the trajectory of the FFO along the $\hat{t}$ axis is contained in the region where the flat space approximation is valid. Therefore, the following procedure is justified.

The quantized scalar field which vanishes for $\hat{x}^1 = 0$ is obtained by simply imposing such a condition on the one without the boundary condition, namely the expression given in (3.9) with unhatted variables replaced by hatted ones. Explicitly, setting the

coefficient of $\cos \hat{p}^1 \hat{x}^1$, which does not vanish for $\hat{x}^1 = 0$, in the expansion $\exp(i\hat{p}^1 \hat{x}^1) = \cos \hat{p}^1 \hat{x}^1 + i \sin \hat{p}^1 \hat{x}^1$, we obtain the relation between the modes

$$\hat{a}_{lm,-\hat{p}^1} = -\hat{a}_{lm\hat{p}^1} . \tag{4.19}$$

This clearly shows that the mode with negative $\hat{p}^1$ is directly related to the mode with positive $\hat{p}^1$ and hence *the number of independent modes is halved* by the imposition of the boundary condition. Intuitively the wave as seen in the hatted frame is reflected perpendicularly by the boundary line. Therefore, the scalar field as quantized by a FFO moving in the direction of the $\hat{t}$ axis and its conjugate momentum are of the form

$$\phi(\hat{t}, \hat{x}^1, \Omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_{-\infty}^{\infty} \frac{d\hat{p}^1}{\sqrt{4\pi E_{k_l \hat{p}^1}}} \left(e^{-iE_{k_l \hat{p}^1} \hat{t}} Y_{lm}(\Omega)(\hat{a}_{lm\hat{p}^1} - \hat{a}_{lm-\hat{p}^1}) + \text{h.c.} \right) \sin \hat{p}^1 \hat{x}^1 . \tag{4.20}$$

$$\hat{\pi}(\hat{t}, \hat{x}^1, \Omega) = \partial_{\hat{t}} \phi(\hat{t}, \hat{x}^1, \Omega)$$
$$= -i \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_{-\infty}^{\infty} \frac{d\hat{p}^1 \sqrt{E_{k_l \hat{p}^1}}}{\sqrt{4\pi}} \left(e^{-iE_{k_l \hat{p}^1} \hat{t}} Y_{lm}(\Omega)(\hat{a}_{lm\hat{p}^1} - \hat{a}_{lm-\hat{p}^1}) - \text{h.c.} \right) \sin \hat{p}^1 \hat{x}^1 , \tag{4.21}$$

where the notation $\hat{\pi}$ reminds us that the conjugate momentum in this frame is defined using the derivative with respect to $\hat{t}$. By using the commutation relation $\left[\hat{a}_{lm\hat{p}^1}, \hat{a}_{l'm'\hat{q}^1}^\dagger \right] = \delta_{ll'} \delta_{mm'} \delta(\hat{p}^1 - \hat{q}^1)$ and the formula $\int_0^\infty d\hat{p}^1 \sin \hat{p}^1 \hat{x}^1 \sin \hat{p}^1 \hat{y}^1 = (\pi/2)(\delta(\hat{x}^1 - \hat{y}^1) - \delta(\hat{x}^1 + \hat{y}^1))$, one can verify that the commutator of the conjugate fields takes the canonical form[17]

$$[\hat{\pi}(\hat{t}, \hat{x}^1, \Omega), \phi(\hat{t}, \hat{y}^1, \Omega')] = -i\delta(\hat{x}^1 - \hat{y}^1)\delta(\cos \theta - \cos \theta')\delta(\varphi - \varphi') . \tag{4.22}$$

4.2.3 Comparison of Hilbert space of FFO and the genuine Minkowski Hilbert space

Let us define for convenience the following combinations of the mode operators:

$$\hat{a}_{lm\hat{p}^1}^\pm \equiv \hat{a}_{lm\hat{p}^1} \pm \hat{a}_{lm,-\hat{p}^1} , \tag{4.23}$$

$$\hat{a}_{lm\hat{p}^1}^{\pm\dagger} \equiv \hat{a}_{lm\hat{p}^1}^\dagger \pm \hat{a}_{lm,-\hat{p}^1}^\dagger \tag{4.24}$$

Cearly, the operators with plus and minus superscripts commute with each other. Then, from the discussion above, the Hilbert space $\mathcal{H}_{\text{FFO}}$ of the FFO in the $(\hat{t}, \hat{x}^1)$ frame is constructed upon the vacuum $|\hat{0}\rangle_-$, defined by

$$\hat{a}_{lm\hat{p}^1}^- |\hat{0}\rangle_- = 0 , \tag{4.25}$$

---

[17]Since $\hat{x}^1$ and $\hat{y}^1$ are both positive, we can discard $-\delta(\hat{x}^1 + \hat{y}^1)$.

by applying the operators $\hat{a}^{-\dagger}_{lm\hat{p}^1}$ repeatedly. In contrast, the genuine Minkowski Hilbert space $\mathcal{H}_M$ is built upon the vacuum $|0\rangle_M$, which is defined to be annihilated by $\hat{a}_{lm\hat{p}^1}$ for all values of l, m and $\hat{p}^1$, by the (repeated) applications of $\hat{a}^{\dagger}_{lm\hat{p}^1}$'s . This means that $\mathcal{H}_M$ can be written as the tensor product

$$\mathcal{H}_M = \mathcal{H}^- \otimes \mathcal{H}^+ , \tag{4.26}$$

where $\mathcal{H}^-$ stands for $\mathcal{H}_{\text{FFO}}$ and the other half $\mathcal{H}^+$ is constructed in the entirely similar manner as for $\mathcal{H}^-$, using the a^+ type operators. From the point of view of FFO, $\mathcal{H}^+$ is unphysical but it is needed for the construction of $\mathcal{H}_M$. Note that this decomposition is completely *different* from the left-right decomposition $\mathcal{H}_M = \mathcal{H}_{W_L} \otimes \mathcal{H}_{W_R}$.

This structure will be important in the discussion of the Unruh-like effect near the horizon of a physical Schwarzschild black hole, to be discussed in Sec. 5.

4.2.4 Quantization by a FFO in a general frame $(\tilde{t}, \tilde{x}^1)$ with the boundary condition

We now consider the quantization by a FFO in a general frame $(\tilde{t}, \tilde{x}^1)$ with the *same boundary condition* $\hat{x}^1 = 0$ along the shock wave. Since this boundary condition is simplest to describe in the $(\hat{t}, \hat{x}^1)$ frame, the most efficient way to quantize in the $(\tilde{t}, \tilde{x}^1)$ frame with such a boundary condition is to express the new conjugate momentum $\tilde{\pi} \equiv \partial_{\tilde{t}}\phi$ in terms of the quantities in the $(\hat{t}, \hat{x}^1)$ frame by applying the relation

$$\frac{\partial}{\partial \tilde{t}} = \frac{\partial \hat{t}}{\partial \tilde{t}} \frac{\partial}{\partial \hat{t}} + \frac{\partial \hat{x}^1}{\partial \tilde{t}} \frac{\partial}{\partial \hat{x}^1} = \gamma \frac{\partial}{\partial \hat{t}} + \gamma\beta \frac{\partial}{\partial \hat{x}^1} \tag{4.27}$$

to $\phi(\hat{t}, \hat{x}^1, \Omega)$, which we already have. Because $\partial_{\tilde{t}}$ contains the spatial derivative $\partial_{\hat{x}^1}$ as well, this leads to an important non-trivial change in the conjugate momentum, however. The result is[18]

$$\tilde{\pi}(\tilde{t}, \tilde{x}^1, \Omega) = -i\widetilde{\mathcal{N}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_0^{\infty} \frac{d\hat{p}^1}{\sqrt{4\pi E_{k_l\hat{p}^1}}} \left[\left((-i\gamma\hat{E}_{k_l\hat{p}^1})e^{-iE_{k_l\hat{p}^1}\hat{t}}Y_{lm}(\Omega)a_{lm\hat{p}^1} + \text{h.c.} \right) \sin\hat{p}^1\hat{x}^1 \right.$$
$$\left. + \gamma\beta\hat{p}^1 \left(e^{-iE_{k_l\hat{p}^1}\hat{t}}Y_{lm}(\Omega)a_{lm\hat{p}^1} + \text{h.c.} \right) \cos\hat{p}^1\hat{x}^1 \right], \tag{4.28}$$

where we have denoted the normalization constant in this frame as $\widetilde{\mathcal{N}}$.

Now let us compute the *equal $\tilde{t}$* canonical commutation relation between $\tilde{\pi}$ and $\tilde{\phi}$. In this process we need to take into account the following two points:

---

[18]To get the explicit form of $\tilde{\pi}(\tilde{t}, \tilde{x}^1, \Omega)$, we must rewrite all the hatted quantities in terms of the tilded ones obtained by the Lorentz transformation (4.16). This produces a rather involved expression. The procedure adopted here can avoid this complication.

(i) Since $\tilde{t} = \gamma(\hat{t} - \beta\hat{x}^1)$, equal $\tilde{t}$ is equivalent to equal $\hat{t} - \beta\hat{x}^1$. In other words, if we denote the hatted time that appears in $\tilde{\pi}$ given by (4.28) by $\hat{t}$ and the one in ϕ by $\hat{t}'$, then the equal $\tilde{t}$ can be expressed as $\hat{t} - \beta\hat{x}^1 = \hat{t}' - \beta\hat{y}^1$, where $\hat{x}^1$ and $\hat{y}^1$ are the spatial coodinates that appear in $\tilde{\pi}$ and ϕ respectively. Therefore we have the important relation

$$\hat{t} - \hat{t}' = -\beta(\hat{x}^1 - \hat{y}^1) \quad \text{at equal } \tilde{t}. \tag{4.29}$$

(ii) The second point to keep in mind is that from the Lorentz transformation we easily find

$$\hat{x}^1 - \hat{y}^1 = \gamma(\tilde{x}^1 - \tilde{y}^1) \quad \text{at equal } \tilde{t}, \tag{4.30}$$

so that the difference in the spatial coodinates in the hatted frame can be rewritten as the rescaled difference in the tilded frame.

With these facts in mind, the equal $\tilde{t}$ commutator $[\tilde{\pi}, \phi]$ is given by

$$\begin{aligned}
&\left[\tilde{\pi}(\tilde{t}, \tilde{x}^1, \Omega), \phi(\tilde{t}, \tilde{y}^1, \Omega') \right] \\
&= -i\widetilde{\mathcal{N}}^2 \sum_{l,m} \sum_{l'm'} \int_0^\infty d\hat{p}^1 \int_0^\infty d\hat{q}^1 \frac{Y_{lm}(\Omega)Y_{l'm'}^*(\Omega')}{4\pi\sqrt{\hat{E}_{k_l\hat{p}^1}\hat{E}_{k_{l'}\hat{q}^1}}} \\
&\quad \left[(-i\hat{E}_{k_l\hat{p}^1}\gamma)\left(e^{-i\hat{E}_{k_l\hat{p}^1}\hat{t}+i\hat{E}_{k_{l'}\hat{q}^1}\hat{t}'} + \text{h.c.}\right)\left[\hat{a}_{lm\hat{p}^1}, \hat{a}_{l'm'\hat{q}^1}^\dagger\right] \sin\hat{p}^1\hat{x}^1 \, \sin\hat{q}^1\hat{y}^1 \right. \\
&\quad \left. + \gamma\beta\hat{p}^1\left(e^{-i\hat{E}_{k_l\hat{p}^1}\hat{t}+i\hat{E}_{k_{l'}\hat{q}^1}\hat{t}'} - \text{h.c.}\right)\left[\hat{a}_{lm\hat{p}^1}, \hat{a}_{l'm'\hat{q}^1}^\dagger\right] \cos\hat{p}^1\hat{x}^1 \, \sin\hat{q}^1\hat{y}^1 \right] \\
&= C_1 + C_2 \, , \tag{4.31}
\end{aligned}$$

where

$$C_1 = -\gamma\widetilde{\mathcal{N}}^2 \sum_{l=0}^\infty \sum_{m=-l}^l \int_0^\infty \frac{d\hat{p}^1}{4\pi} Y_{lm}(\Omega)Y_{lm}^*(\Omega')\left(e^{i\beta\hat{E}_{k_l\hat{p}^1}(\hat{x}^1-\hat{y}^1)} + \text{h.c.}\right)\sin\hat{p}^1\hat{x}^1 \, \sin\hat{p}^1\hat{y}^1 \, , \tag{4.32}$$

$$C_2 = -i\gamma\beta\widetilde{\mathcal{N}}^2 \sum_{l=0}^\infty \sum_{m=-l}^l \int_0^\infty \frac{d\hat{p}^1}{4\pi\hat{E}_{k_l\hat{p}^1}} Y_{lm}(\Omega)Y_{lm}^*(\Omega')\left(e^{i\beta\hat{E}_{k_l\hat{p}^1}(\hat{x}^1-\hat{y}^1)} - \text{h.c.}\right)\cos\hat{p}^1\hat{x}^1 \, \sin\hat{p}^1\hat{y}^1 \, . \tag{4.33}$$

Note that for the the exponents involving $\hat{E}_{k_l\hat{p}^1}$ we used the relation (4.29). The sum over m can be done by the well-known addition theorem for Y_{lm} namely

$$\sum_{m=-l}^l Y_{lm}(\Omega)Y_{lm}^*(\Omega') = \frac{2l+1}{4\pi}P_l(\hat{n}\cdot\hat{n}') \, , \tag{4.34}$$

where $P_l(x)$ is the Legendre polynomial and $\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ denotes the unit vector corresponding to the pair of angles $\Omega(\theta, \varphi)$. Furthermore, the integral over

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$\hat{p}^1$, after rewriting the product of trigonometric functions into a sum of them, can also be performed using the formulas 3.961 of [45] (with the aid of the relation $\partial_z K_0(z) = -K_1(z)$)

$$\int_0^\infty e^{-b\sqrt{k^2+x^2}} \cos axdx = \frac{bk}{\sqrt{a^2+b^2}} K_1(k\sqrt{a^2+b^2}), \tag{4.35}$$

$$\int_0^\infty \frac{x}{\sqrt{k^2+x^2}} e^{-b\sqrt{k^2+x^2}} \sin axdx = \frac{ak}{\sqrt{a^2+b^2}} K_1(k\sqrt{a^2+b^2}), \tag{4.36}$$

where $K_1(z)$ is the Macdonald function (*i.e.* one of the modified Bessel functions) of order 1 and the both formulas are valid for $\mathrm{Re}\, b > 0$, $\mathrm{Re}\, k > 0$. In particular, the convergence condition $b > 0$ is important since in our case, $b = \pm i\beta(\hat{x}^1 - \hat{y}^1)$ and are pure imaginary. Thus, we must regularize them by introducing an infinitesimal positive parameter $\eta > 0$ and replace b by

$$b_- \equiv -i\beta(\hat{x}^1 - \hat{y}^1 + i\eta), \tag{4.37}$$

$$b_+ \equiv +i\beta(\hat{x}^1 - \hat{y}^1 - i\eta). \tag{4.38}$$

Since the rest of the calculations are somewhat tedious but more or less straightforward, we shall describe some intermediate steps in the Appendix D and only list here the important structures that one will encounter as one proceeds.

- The terms which contain $\cos(\hat{x}^1 + \hat{y}^1)$ and $\sin(\hat{x}^1 + \hat{y}^1)$ produced from the product of sines and cosines turn out to cancel completely due to the fact that $\hat{x}^1 + \hat{y}^1$ is positive and generically finite and the regulator η after performing the integrals can be ignored compared to them.

- On the other hand, for the terms containing the difference $\hat{x}^1 - \hat{y}^1$, there are two cases. When the difference is finite and hence η in $b_\pm$ can be ignored, all the terms cancel just as in the case above and hence the commutator vanishes.

  In contrast, when the difference is of order η or smaller, then the contribution remains and becomes proportional to the structure $K_1(\alpha k_l \eta)$, with a finite constant α. Now if we first make a cut-off on the angular momentum l so that $k_l = \sqrt{l(l+1)}/2M$ can be large but finite, then $\alpha k_l \eta \to 0$ as we send $\eta \to 0$. Then, from the behavior of $K_1(z)$ for small z, *i.e.* $K_1(z) \simeq 1/z$, we see that the contribution diverges like $1/\eta$. Thus, we see that as $\hat{x}^1 - \hat{y}^1 \to 0$, the commutator diverges as we remove the regulator.

  Together, this is nothing but the behavior of the δ-function $\delta(\hat{x}^1 - \hat{y}^1)$ which is proportional to $\delta(\tilde{x}^1 - \tilde{y}^1)$ due to the relation (4.30).

- In the other limit where l becomes so large that $k_l \eta$ is large, then, $K_1(z)$ damps like $\sim e^{-z}/\sqrt{z}$ and such a region does not contribute. This indicates that we can effectively replace k_l by a large constant independent of l.

- Then, we are left with the sum over l, which produces the angular δ-functions in the manner

$$\sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\vec{n} \cdot \vec{n'}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\Omega) Y_{lm}^*(\Omega')$$

$$= \delta(\cos\theta - \cos\theta')\delta(\varphi - \varphi') = \frac{1}{2\pi}\delta(\vec{n} \cdot \vec{n'} - 1). \qquad (4.39)$$

Combining, we find that the commutator is proportional to the desired product of δ-functions and the quantization for an arbitrary FFO with the boundary condition along $\hat{x}^1 = 0$ in the vicinity of the horizon is achieved.

Several remarks are in order:

- Although the correct δ-function structure for the canonical commutation relation is confirmed, unfortunately we cannot compute the exact normalization constant because the relevant integrals and the infinite sum cannot be performed exactly. This is regrettable since such a constant must become singular as we let the almost lightlike trajectory approach exactly lightlike, and it would be interesting to see how this comes about.

- Nevertheless, the fact that the quantization for a general FFO with the boundary condition $\phi(\hat{x}^1 = 0) = 0$ can be carried out as we have shown shows that the number of modes that the FFO sees as he/she passes the horizon *does not change* and is naturally *half as many as for the case of the two-sided black hole.*

4.3 Quantization of the scalar field by the observer in the W_F frame

We now consider the quantization in the W_F frame with the same boundary condition along the slightly time-like line, namely $(1 - \epsilon)t_M + x^1 = 0$. Expressed in terms of the W_F variables (see (2.8)), this becomes

$$z_F(e^{t_F} - \epsilon \cosh t_F) = 0. \qquad (4.40)$$

Since z_F need not vanish, we should set $e^{t_F} = \epsilon \cosh t_F$. This can be easily solved for t_F as

$$t_F \simeq \frac{1}{2} \ln \frac{\epsilon}{2}, \qquad (4.41)$$

which is very large and negative.

Now we impose the vanishing condition for ϕ^F along this line. An advantageous feature of the W_F region is that such a line is, practically, contained entirely in the flat region.

Therefore we can make use of the expression in the flat spacetime and the boundary condition on the field $\phi^F(t_F, z_F, \Omega)$ can be written as

$$0 = \int_{-\infty}^{\infty} d\omega \frac{e^{\pi\omega/2}}{4\sqrt{2}\pi} \sum_{l,m} Y_{lm}(\Omega) \left(H_{i\omega}^{(2)}(k_l z_F) e^{-i\omega t_F} a_{lm\omega}^F + \text{h.c.} \right) , \qquad (4.42)$$

where for t_F the value given in (4.41) should be substituted. Evidently, the factor $e^{-i\omega t_F}$ is oscillating extremely rapidly and this is expected to suppress the integral over ω. In order to estimate this effect, we need to know the behavior of $H_{i\omega}^{(2)}(k_l z_F)$ for large ω. Unfortunately, this has not been understood for the Bessel functions of imaginary order. What is known in the literature [44] is the behavior of $H_{i\omega}^{(2)}(\omega y)$ for large ω with fixed y. Just for a guess, if we take y to be very small so that ωy can be regarded as "finite" compare to ω itself in such a formula, it gives the estimate

$$|H_{i\omega}^{(2)}(z)| \lesssim \frac{1}{\sqrt{\omega}} e^{-\omega\pi/2} . \qquad (4.43)$$

This means that the $e^{\pi\omega/2}$ factor in (4.42) is canceled but still the integral is divergent like $\sim \int d\omega/\sqrt{\omega} \sim \sqrt{\omega}$ for larte ω.

In any case, regardless of whether the above estimate is fully reliable or not, let us consider the states for which $|\omega|$ can be large but finite. In other words, we discard states with very high momenta by placing a cut-off in the ω-integral. Then the integral, with the highly oscillating factor $e^{-i\omega t_F}$ removed, is absolutely convergent. Then, with the factor $e^{-i\omega t_F}$ reinstated, we can invoke the Riemann-Lebegues lemma to conclude that as $\epsilon \to 0$ the ω-integral vanishes and the boundary condition (4.42) is automatically fulfilled without requiring any relation between $a_{lm\omega}^F$ and $a_{lm\omega}^{F\dagger}$.

Thus, the conclusion is that, *except for highly excited states*, the boundary condition does not place any relations among the modes and hence the number of independent modes observed in the W_F frame is not halved but is the *same as in the case for the two-sided black hole.* As for the highly excited states, we cannot make definite assertion without the detailed knowledge of the asymptotic behavior of the function $H_{i\omega}^{(2)}(z)$ as $\omega \to \infty$ with z fixed.

4.4 Quantization of the scalar field by the oberserver in the W_R frame

Finally, let us consider the quantization by the observer in the W_R frame.

If we take the same special slightly timelike boundary line which goes through the origin of the coordinate frame (t_M, x^1), this line is outside of the region W_R. Therefore, there is no boundary condition to impose and the modes which exist for the two-sided case are all present and independent.

Although it is a valid argument, it certainly depends crucially on the special choice of the boundary line. Therefore we should also consider the case where the boundary line is slightly shifted to the positive x^1 direction so that it passes inside W_R very close to its lightlike boundary. Explicitly, the boundary line is now taken to be along

$$t = -\frac{1}{1-\epsilon}x^1 + \delta\,, \tag{4.44}$$

where δ is a very small shift. In this case, the imposition of the boundary condition is meaningful and the argument to follow is of more general validity.
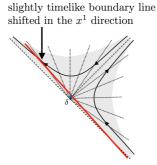

Figure 4.3: A slightly timelike boundary line (shown in red), which is shifted infinitesimally in the positive x^1 direction compared to Figure 4.2.

As discussed in Sec. 3.5, the expansion of $\phi^R(z_R, t_R, \Omega)$ near the horizon is given by[19]

$$\phi^R(z_R, t_R, \Omega) = \int_0^\omega d\omega N_\omega \sum_{l,m} Y_{lm}(\Omega) \left(K_{i\omega}(k_l z_R)e^{-i\omega t_R}a^R_{lm\omega} + \text{h.c} \right)\,, \tag{4.45}$$

$$N_\omega = \frac{\sqrt{\sinh \pi\omega}}{\pi}\,. \tag{4.46}$$

Since (4.44) can be rewritten as $x^+ = z_R e^{t_R} = \epsilon t + \delta$, z_R is small along the boundary line for finite t_R. Now to make use of the form of the $K_{i\omega}(z)$ for small z, we make a cut-off for the angular momentum l and consider the states for which k_l is bounded. Then, we can use the behavior of $N_\omega K_{i\omega}(y)$ for small y, which is given by

$$N_\omega K_{i\omega}(y) \sim \frac{\sqrt{\sinh \pi\omega}}{\pi} \times 2\sqrt{\frac{\pi}{\omega \sinh \pi\omega}} \cos\left(\omega \ln\frac{y}{2} - \arg \Gamma(i\omega)\right)$$

$$= \frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{\omega}}\cos\left(\omega \ln\frac{y}{2} - \arg \Gamma(i\omega)\right)\,. \tag{4.47}$$

Note that this oscillates wildly as y becomes small. y here is $k_l z_R$ and as we send the regulator ϵ to zero, it becomes of order δ, which is very small. Then, just as in the case of the quantization in W_F, by imposing a cut-off for the ω-integral, we can apply the

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[19]Due to the use of the spherical harmonics instead of the plane wave in the transverse directions, the normalization factor N_ω is slightly different from the one in (2.31).

Riemann-Lebesgue lemma to conclude that the Fourier type integral tends to vanish and the boundary condition $\phi^R = 0$ is automatically satisfied along the matter trajectory.

Therefore, the conclusion should be the same as in the case of the W_F observer: If we exclude the the highly excited states, the boundary condition does not impose any relations to the mode operators and the degrees of freedom remain the same as in the two-sided case. For the energetic states, more exact computation is needed to make definite statements.

5   Implications on the quantum equivalence principle, the firewall phenomenon and the Unruh effect

Having analyzed and compared the quantization of a scalar field by different natural observers in a concrete manner, we now consider the implications of our results.

5.1   Quantum equivalence principle and the firewall phenomenon

One of the clear results is that the degrees of freedom of the modes that the observer sees are in general different, both for the case of the two-sided eternal black hole and the more physical one-sided one. Explicitly, the FFO and the W_F observers see the same number of modes, while the observer in the W_R frame finds half as many as the above two. This property is common to the two-sided and the one-sided cases. On the other hand, due to the imposition of the boundary condition, which expresses the effect of the collapsing matter, the degrees of freedom of the modes for the one-sided case are halved for each type of observer compared with the two-sided case[20].

The fact that the size of the quantum Hilbert space is halved for W_R observer is natural since such an observer can only see a part of the spacetime due to the presence of the horizon for him/her.

Whether the equivalence principle holds quantum mechanically is quite a different question. It asks whether the FFO upon crossing the "horizon", which does not exist for him/her *classically*, sees extra or less degrees of freedom of the *quantum* excitation modes of a field. Our explicit computation shows that for both the eternal and the physical black holes *the quantum equivalence principle holds naturally.* This is essentially due to the fact that no new boundary conditions for the scalar field appear as seen by a FFO who goes through the "horizon".

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[20]To be precise, as described in Sec.4. 3 and 4. 4, for the W_F and W_R observers, we cannot make a completely definite statement on the highly excited states about the effect of the boundary condition for technical reason.

Needless to say, this conclusion is valid under the assumption that the metric of the interior of the Schwarzschild black hole is essentially given by the Vaidya type metric. If the interior of the black hole is such that it cannot be specified just by the information of the metric, the conclusion may differ. However, as long as the classical Schwarzschild black holes produced by the collapse of matter is concerned, our assumption is conservative and should be reasonable.

Thus, for a large enough black hole which itself can be treated classically, our explicit computations for the quantum effects of the massless scalar field as seen by the three types of observers should be reliable and in particular the freely falling observer does not encounter the so-called firewall phenomenon.

5.2　Unruh-like effect near the horizon of a physical black hole

The Unruh effect [15] is the simplest example of the non-trivial quantum phenomena due to the difference of the vacua for the relatively accelerated observers. In the original case treated by Unruh, a Rindler observer uniformly accelerated in the flat Minkowski space in the positive x^1 direction with accleration a (confined to the wedge W_R), sees in the Minkowski vacuum $|0\rangle_M$ a swarm of particles of energy ω with the number density distribution given by

$$\frac{\langle N_\omega^R \rangle}{\text{Vol.}} = \frac{_M\langle 0|a_\omega^{R\dagger} a_\omega^R|0\rangle_M}{_M\langle 0|0\rangle_M \text{Vol.}} \propto \frac{1}{e^{2\pi\omega/a} - 1} \,. \tag{5.1}$$

Evidently, this coincides with the thermal distribution of bosons at temperature $a/2\pi$. In fact this computation is *truly thermal in nature* since $|0\rangle_M$ is an entangled state consisting of the states of W_L as well as of W_R, and one must take a trace over all the states of W_L to obtain the distribution above.

Although in this example the background is taken to be the flat spacetime to begin with, one might expect a similar phenomenon to be seen by the stationary observer just outside the horizon of a physical Schwarzschild black hole, since the spacetime there is well-approximated by the right Rindler wedge of a flat Minkowski space.

However, the analysis cannot be the same for the following reasons. First, there is no W_L region for the one-sided black hole and hence whatever distribution we obtain is *not* truly thermal in nature. It simply shows that the concept of "a particle" depends crucially on the vacuum state, even if it is a pure state. The second reason is the fact that, although the region of our interest is locally a flat Minkowski space, we must take into account the effect of the boundary condition for the scalar field and its vacuum seen the FFO, who corresponds to the Minkowski observer in the Unruh's set up. As discussed in Sec.4.2.3, however, the vacuum $|\hat{0}\rangle_-$ for the FFO is *not* the genuine Minkowski vacuum.

A related difference is that, as discussed in Sec.4.4, in our setup the scalar field in the W_R frame is not affected by the boundary condition and hence the number of modes seen in that frame is the *same* as that of FFO. This is in contrast to the case of the flat space, where the number of modes for the W_R observer is *half* that of the Minkowski observer.

Thus, the question of interest is what the distribution of the W_R particles is in the vacuum of the FFO. To answer this question, we must express the mode operators $a_{lm\omega}^R$ and their conjugates of the W_R observer in terms of the field $\phi(\hat{t}, \hat{x}^1, \Omega)$ and its modes of a FFO[21].

Unfortunately, in general this computation cannot be performed accurately due to the lack of our knowledge of the fields outside the approximately flat region. The required calculation is of the form

$$a_{lm\omega}^R = i \int d\varphi \int \sin\theta d\theta Y_{lm}^*(\Omega) \int_0^\infty \frac{dz_R}{z_R} f_{\omega,l}^*(t_R, z_R) \overleftrightarrow{\partial_{t_R}} \phi^{\mathrm{FFO}}(\hat{t}, \hat{x}^1, \Omega), \qquad (5.2)$$

where $f_{\omega,l}(t_R, z_R) Y_{lm}(\Omega)$ is the solution of the equation of motion, corresponding to the mode $a_{lm\omega}^R$, in the right Rindler wedge in the background of the Schwarzschild black hole. What we only know is that this function takes the form $f_{\omega,l}(t_R, z_R) \simeq N_\omega K_{i\omega}(|k_l|z_R)e^{-i\omega t_R}$ in the approximately flat region where $z_R \lesssim M$. Therefore, integration over z_R, which extends outside such a region, cannot be performed explicitly.

There are, however, a class of modes for which the computation can be performed sufficiently accurately by the use of the function for the flat space region. These are the ones with large anglular momentum l such that $|k_l|M = \sqrt{l(l+1)}/2 \gg 1$. To see this, let us expand the scalar field into angular momentum eigenstates as $\phi(t, r, \Omega) = \sum_{l,m} \phi_{lm}(r) Y_{lm}(\Omega)$ and write down the equation of motion for $\phi_{lm}(t, r)$ in the Schwarzschild metric. It is given by

$$0 = -\frac{1}{1 - \frac{2M}{r}}\partial_t^2 \phi_{lm} + \frac{2(r-M)}{r^2}\partial_r \phi_{lm} + \left(1 - \frac{2M}{r}\right)\partial_r^2 \phi_{lm} - \frac{l(l+1)}{r^2}\phi_{lm}. \qquad (5.3)$$

Now we look at the region $r \gg M$ where the flat space approximation is no longer valid. In such a region, writing $\phi_{lm}(t, r) = e^{\pm i\omega t}\tilde{\phi}_{lm}(r)$, the equation for $\tilde{\phi}_{lm}(r)$ simplifies to

$$0 = \left(\omega^2 + \frac{2}{r}\partial_r + \partial_r^2 - \frac{l(l+1)}{r^2}\right)\tilde{\phi}_{lm}(r). \qquad (5.4)$$

The solution is well-known and is given, with a certain normalization, by

$$\tilde{\phi}_{lm}(r) = \sqrt{\frac{\omega}{r}}\, J_{l+\frac{1}{2}}(\omega r), \qquad (5.5)$$

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[21]The reason for focusing on the FFO in the $(\hat{t}, \hat{x}^1)$ frame is simply that the effect of the boundary condition is the simplest in such a frame. For the other frames of FFO, one can make a Lorentz transformation for the FFO, *with the boundary condition kept intact.*

where $J_{l+\frac{1}{2}}(\omega r)$ is the Bessel function. Its asymptotic form for large l can be obtained from the formula 10.19.1 of [46] as

$$J_{l+\frac{1}{2}}(\omega r) \sim \frac{1}{\sqrt{(2l+1)\pi}} e^{-(l+\frac{1}{2})\ln\frac{2l+1}{e\omega r}} \,. \tag{5.6}$$

This shows that for $l \gtrsim \omega r$, this expression is exponentially small in l and contributes negligibly to the integral over z_R. *Thus, for such modes with high angular momenta, we can effectively need only the function in the flat region* and the computation is possible. Such a calculation is at the same time self-consistent because $K_{i\omega}(|k_l|z_R)$ damps exponentially for large $|k_l|z_R$ and for large enough l this quantity is already large for $z_R \simeq M$. Therefore, contribution from $z_R \gtrsim M$ region is safely neglected.

To perform the computation of (5.2), first consider the projection of the angular part in (5.2) with the use of the orthogonality of the spherical harmonics. Since ϕ^{FFO} contains both Y_{lm} and Y_{lm}^*, the relevant formulas are $\int d\Omega Y_{lm}^*(\Omega)Y_{l'm'}(\Omega) = \delta_{ll'}\delta_{mm'}$ and $\int d\Omega Y_{lm}^*(\Omega)Y_{l'm'}^*(\Omega) = (-1)^m\delta_{ll'}\delta_{m,-m'}$. (The second formula follows from the first by using the relation $Y_{lm}^* = (-1)^m Y_{l,-m}$.)

Therefore, after the removal of the angular part, what we need to compute is

$$a^R_{lm\omega} = i\int_0^\infty \frac{dz_R}{z_R} N_\omega K_{i\omega}(|k_l|z) e^{i\omega t_R}\overleftrightarrow{\partial_{t_R}} \int_{-\infty}^\infty \frac{d\hat{p}^1}{\sqrt{4\pi E_{k_l\hat{p}^1}}} \left(e^{-iE_{k_l\hat{p}^1}\hat{t}}\hat{a}^-_{lm\hat{p}^1} \right.$$
$$\left. + (-1)^m e^{iE_{k_l\hat{p}^1}\hat{t}}\hat{a}^{-\dagger}_{l,-m\hat{p}^1} \right) \sin \hat{p}^1\hat{x}^1 \,. \tag{5.7}$$

To perform the differentiation with respect to t_R, we must use the relation between $\hat{t}$ and t_R given by the Lorentz transformations

$$\hat{t} = \hat{\gamma}(t_M + \hat{\beta}x^1) = \hat{\gamma}z_R(\sinh t_R + \hat{\beta}\cosh t_R) \,, \tag{5.8}$$

$$\hat{x}^1 = \hat{\gamma}(x^1 + \hat{\beta}t_M) = \hat{\gamma}z_R(\cosh t_R + \hat{\beta}\sinh t_R) \,, \tag{5.9}$$

$$\hat{\beta} = 1 - \epsilon \equiv \tanh\xi \,, \qquad \hat{\gamma} = \frac{1}{\sqrt{1-\beta^2}} \equiv \cosh\xi \,, \tag{5.10}$$

where we introduced the rapidity variable ξ. Then, the relevant part of (5.7) can be computed as

$$e^{i\omega t_R}\overleftrightarrow{\partial_{t_R}} \left[e^{-iE_{k_l\hat{p}^1}\hat{t}}\sin(\hat{p}^1\hat{x}^1) \right]$$
$$= e^{i\omega t_R}e^{-iaz_R}\left[-i(a'z_R + \omega)\sin bz_R + b'z_R\cos bz_R \right]$$
$$= -\frac{1}{2}(a'z + \omega)e^{i\omega t}\left(e^{-i(a-b)z} - e^{-i(a+b)z} \right) + \frac{1}{2}b'ze^{i\omega t}\left(e^{-i(a-b)z} + e^{-i(a+b)z} \right) \,, \tag{5.11}$$

where

$$a \equiv E_{k_l\hat{p}^1}\hat{\gamma}(\sinh t_R + \hat{\beta}\cosh t_R) \,, \qquad b \equiv \hat{p}^1\hat{\gamma}(\cosh t_R + \hat{\beta}\sinh t_R) \,, \tag{5.12}$$

$$a' \equiv E_{k_l\hat{p}^1}\hat{\gamma}(\cosh t_R + \hat{\beta}\sinh t_R) \,, \qquad b' \equiv \hat{p}^1\hat{\gamma}(\sinh t_R + \hat{\beta}\cosh t_R) \,. \tag{5.13}$$

These expressions can be further simplified by introducing the parametrization

$$E_{k_l \hat{p}^1} = |k_l| \cosh \hat{u}\,, \qquad \hat{p}^1 = |k_l| \sinh \hat{u}\,. \tag{5.14}$$

Then, we can write

$$a \pm b = |k_l| \sinh \rho_\pm\,, \qquad a' \pm b' = |k_l| \cosh \rho_\pm\,, \tag{5.15}$$

$$\rho_\pm \equiv \xi + t_R \pm \hat{u}\,. \tag{5.16}$$

Now we consider the integral over z_R in (5.7). The basic integrals we need are $A_1(c,k)$ and $A_2(c,k)$ given in (B.24) and (B.25) in Appendix B.2.1. Specifically, the ones we need are with $c = \pm(a \pm b)$ and $k = |k_l|$. When these parameters are substituted the integrals simplify drastically and we obtain

$$A_1(a \pm b, |k_l|) = C_\omega \left(e^{\pi\omega/2} e^{-i\omega\rho_\pm} + e^{-\pi\omega/2} e^{i\omega\rho_\pm} \right)\,, \tag{5.17}$$

$$A_2(a \pm b, |k_l|) = \frac{\omega C_\omega}{|k_l| \cosh \rho_\pm} \left(e^{\pi\omega/2} e^{-i\omega\rho_\pm} - e^{-\pi\omega/2} e^{i\omega\rho_\pm} \right)\,, \tag{5.18}$$

$$A_1(-(a \pm b), |k_l|) = C_\omega \left(e^{\pi\omega/2} e^{i\omega\rho_\pm} + e^{-\pi\omega/2} e^{-i\omega\rho_\pm} \right)\,, \tag{5.19}$$

$$A_2(-(a \pm b), |k_l|) = \frac{\omega C_\omega}{|k_l| \cosh \rho_\pm} \left(e^{\pi\omega/2} e^{i\omega\rho_\pm} - e^{-\pi\omega/2} e^{-i\omega\rho_\pm} \right)\,, \tag{5.20}$$

where

$$C_\omega \equiv \frac{\pi}{2\omega \sinh \pi\omega}\,. \tag{5.21}$$

Further, it is convenient to use the rapidity-based oscillators

$$\hat{a}_{lm\hat{u}}^- \equiv \sqrt{E_{k_l \hat{p}^1}}\, \hat{a}_{lm\hat{p}^1}^-\,, \tag{5.22}$$

similarly to (2.51).

Now, using these formulas, it is straightforward to compute the RHS of the formula (5.7) and get the form of $a_{lm\omega}^R$ (and its conjugate) in terms of the FFO mode operators $\hat{a}_{lm\rho}^-$ and $\hat{a}_{lm\rho}^{-\dagger}$. The answers take rather simple forms:

$$a_{lm\omega}^R = \frac{1}{2i} \frac{e^{-i\omega\xi}}{\sqrt{\pi \sinh \pi\omega}} \int_{-\infty}^\infty d\rho \sin \omega\rho \left(e^{\pi\omega/2} \hat{a}_{lm\rho}^- - (-1)^m e^{-\pi\omega/2} \hat{a}_{l,-m\rho}^{-\dagger} \right)\,, \tag{5.23}$$

$$a_{lm\omega}^{R\dagger} = \frac{1}{2i} \frac{e^{i\omega\xi}}{\sqrt{\pi \sinh \pi\omega}} \int_{-\infty}^\infty d\rho \sin \omega\rho \left((-1)^m e^{-\pi\omega/2} \hat{a}_{l,-m\rho}^- - e^{\pi\omega/2} \hat{a}_{lm\rho}^{-\dagger} \right)\,. \tag{5.24}$$

One can check that they satisfy the correct commutation relation $[a_{lm\omega}^R, a_{l'm'\omega'}^{R\dagger}] = \delta_{ll'} \delta_{mm'} \delta(\omega - \omega')$.

Finally, with the expressions (5.24), we can compute the expectation value of the number operator for the W_R "particles" in the FFO vacuum $|\hat{0}\rangle_-$. The result is

$$\frac{{}_-\langle \hat{0}| a_{lm\omega}^{R\dagger} a_{lm\omega}^R |\hat{0}\rangle_-}{{}_-\langle \hat{0}|\hat{0}\rangle_-} = \frac{1}{e^{2\pi\omega} - 1} \frac{2}{\pi} \int_{-\infty}^\infty d\rho \sin^2 \omega\rho\,. \tag{5.25}$$

Several remarks are in order:

(i) We recognize that the first factor is of the same form as the familiar "thermal" distribution. We emphasize however that in this case it is not genuinely thermal since W_L modes do not exist and hence no tracing over them is involved. The fact that the form looks thermal stems from the fact that the expression of $a_{lm\omega}^R$ in terms of $\hat{a}_{lm\rho}^-$ and its conjugate in (5.23) is essentially the same as (2.53), valid for the entire Minkowski space including the region W_L.

(ii) The last integral represents the cohererent sum over infinite number of rapidities which contribute to the W_R mode. Although it appears to depend on ω, this factor is divergent and depending on how we cut it off, the ω-dependence will be different. Moreover, as it becomes clear from the comparison with the usual Unruh effect below, this factor comes from the nature of the boundary condition along the shock wave, $i.e.$ it depends on the interaction between the falling matter and the scalar field. Therefore, this integral is ambiguous and the form of its ω dependence should not be taken seriously. It indicates, however, that an extra ω-dependence, other than the usual thermal factor, can be possible.

(iii) As the last remark, note that the dependence on ξ, the Lorentz boost parameter, disappeared in the distribution. This is quite natural since the vacuum $|\hat{0}\rangle_-$ should be Lorentz invariant.

In any case, we have found that, even in the case of the one-sided black hole, the Unruh-like effect does exist.

It is instructive to compare this with the case of the usual Unruh effect. From (2.53), it is easy to find

$$
\frac{_M\langle 0|a_{k\omega}^{R\dagger} a_{k\omega}^R|0\rangle_M}{_M\langle 0|0\rangle_M} = \int_{-\infty}^{\infty} \frac{du}{4\pi \sinh \pi\omega} \int_{-\infty}^{\infty} du' e^{-i\omega(u-u')} e^{-\pi\omega} \frac{_M\langle 0|[a_{ku}^M, a_{ku'}^{M\dagger}]|0\rangle_M}{_M\langle 0|0\rangle_M}
$$

$$
= \frac{1}{2\pi(e^{2\pi\omega}-1)} \frac{V_{R2}}{(2\pi)^2} \int_{-\infty}^{\infty} du, \qquad (5.26)
$$

where $\frac{V_{R2}}{(2\pi)^2} = \delta^2(k-k)$ is the volume of the two-dimensional space and the divergent integral $\int_{-\infty}^{\infty} du$ counts all the modes with different rapidities making up a W_R particle wave. Note that in this flat space Unruh effect, the ω dependence $e^{-i\omega(u-u')}$ cancels out due to the appearance of $\delta(u-u')$ coming from the commutator $[a_{ku}^R, a_{ku'}^{R\dagger}]$ and we have exactly the thermal form, as is well-known.

6 Summary and discussions

6.1 Brief summary

In this work, we studied the issue of the observer-dependence for the quantization of fields in a curved spacetime, which is one of the crucial problems that one must deal with whenever one discusses quantum gravity. Understanding of this issue is particularly important in cases where an event horizon exists for some of the observers. Explicitly, we have focused on the quantization of a scalar field in the most basic such configuration, namely the spacetime in the vicinity of the horizon of the four dimensional Schwarzschild black hole, including the interior as well as the exterior. Detailed and comprehensive analyses are performed for the three typical observers and clarified how the modes they observe are related. We studied both the two-sided eternal case and the more physical one-sided case produced by the falling shell, or a shock wave. For the latter, the effect of the collapsing matter upon the scalar field outside of the shell is represented by an effective boundary condition along the shock wave.

One important conclusion obtained from such explicit calculations is that the free-falling ovserver sees no change in the Hilbert space structure of the quantized field as he/she crosses the horizon. In other words, the equivalence principle holds quantum mechanically as well, at least in the above sense.

Another result worth emphasizing is that in the one-sided case despite the fact that there are no counterpart of the left-Rindler modes in the vacuum of the freely-falling observer and hence no tracing procedure over them is relevant, there still exists a Unruh-like effect. Namely, in such a vacuum the number density of the W_R modes contains the universal factor of "thermal" distribution in the frequency ω (apart from a divergent piece which depends on the interaction between the scalar field and the falling matter.)

Besides these results, comprehensive and explicit knowledge of the properties and the relations of the Hilbert spaces for the different observers have been obtained and we believe this will be of use in better understanding of quantum properties of gravitational physics.

6.2 Discussions

Evidently, the problem of observer dependence that we studied in the semi-classical regime in this work is of universal importance in any attempt to understand quantum gravity. In particular, it would be extremely interesting to see how this problem appears and should be treated in the construction of the "bulk" from the "boundary" in the AdS/CFT

correspondence, which is anticipated to give important hints for formulating quantum gravity and understanding quantum black hole. Although there have been some attempts to address this question, it is not well-understood how the change of frame (*i.e.* the choice of "time") for the quantization, both in the bulk and the boundary, is expressed and controled in the AdS/CFT context. The best place to look into would be the AdS$_3$/CFT$_2$ setting, where at least we have some knowledge of how the structure of CFT$_2$ changes under a re-definition of "time " by a conformal change of variable [47, 48]. A further advantage to explore the observer dependence in AdS$_3$/CFT$_2$ is that AdS$_3$ black holes (*i.e.* , BTZ black holes) are locally equivalent to the pure AdS$_3$ spacetime and we can solve the equations of motion in the black hole spacetime in the same manner as for the pure AdS$_3$.

In this work, we have concentrated on the relations between the modes seen by different observers and have not touched upon the correlation functions between the fields, especially the ones in different frames. Some correlation functions in the Rindler wedges of the Minkowski space have been studied [49], but the most interesting quenstion of whether one can extract physical information from behind the horizon by quantum means is yet to be answered. We hope to study these and related questions and give a report in the near future.

Acknowledgment

A   Orthogonality and completeness relations for the modified Bessel functions of imaginary order

For various basic computations performed in the main text using the expansions in terms of the eigenmodes, the orthogonality and the completeness of the modified Bessel functions of imaginary order are essential. In this appendix, we give some useful comments on such relations previously obtained in the literature and provide additional information.

A.1   Orthogonality

The orthogonality relations are needed in extracting each mode from the expansion of the scalar field appropriate for various coordinate frames. Such relation for $K_{i\omega}(x)$ is proven

in [39–41] and takes the form

$$\int_0^\infty \frac{dx}{x} K_{i\omega}(x) K_{i\omega'}(x) = \frac{1}{\mu(\omega)} \left(\delta(\omega - \omega') + \delta(\omega + \omega') \right), \tag{A.1}$$

where $\mu(\omega)$ here and below is given by

$$\mu(\omega) \equiv \frac{2\omega \sinh \pi\omega}{\pi^2}. \tag{A.2}$$

The corresponding relations for the Hankel functions $H_{i\omega}^{(i)}$ for $i = 1, 2$ have not been explicitly given in the literature but can be derived without difficulty, for example, by the method described in [41]. The result is

$$\int_0^\infty \frac{dx}{x} H_{i\omega}^{(i)}(x) H_{i\omega'}^{(i)}(x) = \frac{4e^{\eta_i \pi\omega}}{\pi^2 \mu(\omega)} \left(\delta(\omega - \omega') + \delta(\omega + \omega') \right) \quad \eta_1 = +1, \quad \eta_2 = -1. \tag{A.3}$$

A.2 Completeness

The completeness relation for the function $K_{i\omega}(x)$ can be written as

$$\int_0^\infty d\omega \mu(\omega) K_{i\omega}(x) K_{i\omega}(y) = x\delta(x - y), \tag{A.4}$$

Since $K_{-i\omega} = K_{i\omega}$, we can, if we wish, extend the range of integration to $[-\infty \le \omega \le \infty]$ and multiply the RHS by a factor of 2.

This relation is equivalent to the inverse of the so-called Kontorovich-Lebedev (KL) transform [42] below. KL transform $f(\omega, y)$ of a function $g(x, y)$ with respect to x (where y is a parameter) is defined by

$$f(\omega, y) = \mu(\omega) \int_0^\infty \frac{dx}{x} K_{i\omega}(x) g(x, y). \tag{A.5}$$

Then, $g(x, y)$ is obtained in terms of $f(\omega, y)$ by the formula

$$g(x, y) = \int_0^\infty d\omega K_{i\omega}(x) f(\omega, y). \tag{A.6}$$

If we take $g(x, y) = x\delta(x - y)$, then the formula (A.5) gives $f(\omega, y) = \mu(\omega) K_{i\omega}(y)$. Substituting this into (A.6) then gives

$$x\delta(x - y) = \int_0^\infty d\omega \mu(\omega) K_{i\omega}(x) K_{i\omega}(y), \tag{A.7}$$

which is precisely the completeness relation (A.4).

In fact, without resorting to the KL-formula, there is a rather elementary derivation of (A.4), starting from the following integral formula [43]:

$$\int_0^\infty d\omega \cosh a\omega K_{i\omega}(x) K_{i\omega}(y) = \frac{\pi}{2} K_0(\sqrt{x^2 + y^2 + 2xy \cos a}), \tag{A.8}$$

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valid for $x, y > 0$, $|\operatorname{Re} a| + |\arg x| < \pi$. (The second condition is stringent. We cannot set $a = \pi$ from the beginning.) First by differntiating this with respect to a, we get

$$\int_0^\infty d\omega\, \omega \sinh a\omega\, K_{i\omega}(x) K_{i\omega}(y) = \frac{\pi x x' \sin a}{2\sqrt{x^2 + y^2 + 2xy \cos a}} K_1(\sqrt{x^2 + y^2 + 2xy \cos a}),$$

(A.9)

where we used the formula $\partial_z K_0(z) = -K_1(z)$. Now we set $a = \pi - \epsilon$, where ϵ is a positive infinitesimal quantity. Then the RHS becomes

$$\frac{\pi x y \epsilon}{2\sqrt{(x-y)^2 + xy\epsilon^2}} K_1(\sqrt{(x-y)^2 + xy\epsilon^2}).$$

(A.10)

For $x - y \neq 0$, this vanishes as $\epsilon \to 0$, *i.e.* as $a \to \pi$. On the other hand, for small $x - y$, using the small argument expansion $K_1(z) \simeq \frac{1}{z}$, (A.10) becomes

$$\frac{\pi}{2} \frac{\epsilon x y}{(x-y)^2 + \epsilon^2 xy}.$$

(A.11)

By making a rescaling $x \to x/\sqrt{xy}$ and $y \to y/\sqrt{xy}$ in the well-known representation of the delta function, namely, $\delta(x - y) = (\epsilon/\pi)/((x-y)^2 + \epsilon^2))$, we readily obtain

$$\delta((x-y)/\sqrt{xy}) = \sqrt{xy}\delta(x-y) = \frac{1}{\pi} \frac{\epsilon x y}{(x-y)^2 + \epsilon^2 xy}.$$

(A.12)

Comparing with (A.11) we obtain the completeness relation (A.4).

The corresponding completeness relations for the Hankel functions are given by

$$\int_0^\infty d\omega \frac{\pi^2 \mu(\omega)}{4 e^{\eta_i \pi \omega}} H_{i\omega}^{(i)}(x) H_{i\omega}^{(i)}(y) = x\delta(x - y)$$

(A.13)

where the sign η_i is as defined in (A.3).

B   Extraction of the modes in various wedges and their relations

In this appendix, we provide some details of the computations concerning the extraction of the modes and their relations described in Sec. 2 of the main text.

B.1   Klein-Gordon inner products and extraction of the modes

We will be interested in a $d-$dimensional curved space with the metric of the form

$$ds^2 = -N(x)^2 dt^2 + g_{ab} dx^a dx^b,$$

(B.1)

where $N(x)$ is the lapse function. Let f_A, f_B be two independent solutions of the Klein-Gordon equation for this metric. Define the following current

$$J^\mu_{f_A, f_B}(x) \equiv f_A^*(x) \overleftrightarrow{\nabla}^\mu f_B \,, \tag{B.2}$$

which is covariantly conserved $\nabla_\mu J^\mu_{f_A, f_B}(x) = 0$. Let Σ be the constant t surface. The conservation property above means that the Klein-Gordon inner product defined by

$$(f_A, f_B)_{\text{KG}} \equiv i \int_\Sigma d^{d-1}x \frac{\sqrt{g}}{N} n_\mu J^\mu_{f_A, f_B} \,, \tag{B.3}$$

where n^μ is the future directed unit vector normal to Σ, is independen of t. This formula is useful in extracting the modes from the field expressed in various coordinates.

B.1.1 The right Rindler wedge

Hereafter, we will sed $d = 4$. In the right Rindler wedge, the metric is given by

$$ds^2 = -z^2 dt^2 + dz^2 + \sum_{i=2}^3 (dx^i)^2. \tag{B.4}$$

In this case, we can identify $N = z, g_{zz} = 1, g_{ij} = \delta_{ij}, \sqrt{g} = 1$, and hence the Klein-Gordon inner product in the right Rindler wedge is defined as

$$(f_A, f_B)_{\text{KG}}^R = i \int_0^\infty \frac{dz}{z} \int d^2x (f_A^* \overleftrightarrow{\partial_t} f_B) \,. \tag{B.5}$$

The solutions of the Klein-Gordon equation in this coordinate frame are

$$f_{k\omega}^R(t, z, x) = N_\omega^R K_{i\omega}(|k|z) e^{i(kx - \omega t)} \,,$$
$$(N_\omega^R)^2 = \frac{\sinh \pi\omega}{\pi^2 (2\pi)^2} \,. \tag{B.6}$$

Let us compute the Klein-Gordon inner product of such functions explicitly. We get

$$(f_{k\omega}^R, f_{k'\omega'}^R)_{\text{KG}}^R = \int_0^\infty \frac{dz}{z} \int d^2x N_\omega^R N_{\omega'}^R K_{i\omega}(|k|z) K_{i\omega'}(|k'|z)(\omega + \omega') e^{i(k-k')x} e^{-i(\omega-\omega')t}$$

$$= (2\pi)^2 (\omega + \omega') \delta(k - k') N_\omega^R N_{\omega'}^R e^{-i(\omega-\omega')t} \int_0^\infty \frac{dz}{z} K_{i\omega}(|k|z) K_{i\omega'}(|k'|z)$$

$$= \delta(k - k') \delta(\omega - \omega') \,, \tag{B.7}$$

where, getting to the last line, we used the orthogonality of the modified Bessel function (A.1) for $\omega, \omega' > 0$.

Recall that the scalar field in the right Rindler wedge can be expanded as

$$\phi^R(t_R, z_R, x) = \int_0^\infty d\omega \int d^2k \left[f_{k\omega}^R(t_R, z_R, x) a_{k\omega}^R + \text{h.c.} \right] \,. \tag{B.8}$$

The modes $a_{k\omega}^R$ and $a_{k\omega}^{R\dagger}$ are extracted using the Klein-Gordon inner product as

$$a_{k\omega}^R = (f_{k\omega}^R, \phi^R)_{\text{KG}}^R \,, \qquad a_{k\omega}^{R\dagger} = -(f_{k\omega}^{R*}, \phi^R)_{\text{KG}}^R \,. \tag{B.9}$$

B.1.2 The future Rindler wedge

In the future Rindler wedge, the metric is given by

$$ds^2 = -dz_F^2 + z_F^2 dt_F^2 + \sum_{i=2}^{3} (dx^i)^2 \,. \tag{B.10}$$

In this case, z_F is the time variable, t_F is the space variable and $N = 1$, $g_{t_F t_F} = z_F^2$, $g_{ij} = \delta_{ij}$, $\sqrt{g} = z_F$. The Klein-Gordon inner product in the future Rindler wedge is defined as

$$(f_A, f_B)_{\text{KG}}^F = i \int_{-\infty}^{\infty} dt_F \int d^2x z_F (f_A^* \overleftrightarrow{\partial_{z_F}} f_B) \,. \tag{B.11}$$

Then the solutions of the Klein-Gordon equation which damps at large $|k|z_F$ are

$$f_{k\omega}^{(2)}(t_F, z_F, x) = N_\omega^F H_{i\omega}^{(2)}(|k|z_F) e^{i(kx - \omega t_F)} \,,$$
$$(N_\omega^F)^2 = \frac{e^{\pi \omega}}{8(2\pi)^2} \,. \tag{B.12}$$

The Klein-Gordon inner product of these functions is given by

$$
\begin{aligned}
& (f_{k\omega}^{(2)}, f_{k'\omega'}^{(2)})_{\text{KG}}^F \\
&= i \int_{-\infty}^{\infty} dt \int d^2x z N_\omega^F N_{\omega'}^F \left(H_{i\omega'}^{(1)}(|k'|z_F) \partial_{z_F} H_{i\omega}^{(2)}(|k|z_F) - H_{i\omega}^{(2)}(|k|z_F) \partial_{z_F} H_{i\omega'}^{(1)}(|k'|z_F) \right) \\
&\quad \cdot e^{-\pi\omega} e^{i(k-k')x} e^{-i(\omega-\omega')t} \\
&= i(2\pi)^3 \delta(k - k') \delta(\omega - \omega') \\
&\quad \cdot z_F N_\omega^F N_{\omega'}^F \left(H_{i\omega'}^{(1)}(|k'|z_F) \partial_{z_F} H_{i\omega}^{(2)}(|k|z_F) - H_{i\omega}^{(2)}(|k|z_F) \partial_{z_F} H_{i\omega'}^{(1)}(|k'|z) \right) e^{-\pi\omega} \\
&= \delta(k - k') \delta(\omega - \omega') \,.
\end{aligned} \tag{B.13}
$$

To get to the last line, we used the identity

$$H_{i\omega}^{(1)}(|k|z) \partial_z H_{i\omega}^{(2)}(|k|z) - H_{i\omega}^{(2)}(|k|z) \partial_z H_{i\omega}^{(1)}(|k|z) = -i\frac{4}{\pi z} \,. \tag{B.14}$$

By similar manipulations, it is easy to get the following inner products

$$(f_{k\omega}^{(2)*}, f_{k'\omega'}^{(2)*})_{\text{KG}}^F = -\delta(k - k') \delta(\omega - \omega') \,, \qquad (f_{k\omega}^{(2)}, f_{k'\omega'}^{(1)})_{\text{KG}}^F = 0 \,. \tag{B.15}$$

Recall that the scalar field in the future Rindler wedge can be expanded as

$$\phi^F(t_F, z_F, x) = \int_{-\infty}^{\infty} d\omega \int d^2k \left[f_{k\omega}^{(2)}(t_F, z_F, x) a_{k\omega}^F + \text{h.c.} \right] \,. \tag{B.16}$$

Taking the inner product with $f_{k\omega}^{(2)}$, we obtain

$$a_{k\omega}^F = (f_{k\omega}^{(2)}, \phi^F)_{\text{KG}}^F \,, \qquad a_{k\omega}^{F\dagger} = -(f_{k\omega}^{(2)*}, \phi^F)_{\text{KG}}^F \,. \tag{B.17}$$

B.2 Mode operators of W_R and W_F in terms of those of the Minkowski space-time

B.2.1 Useful integrals involving $K_{i\omega}(z)$

We shall first derive several useful integrals involving $K_{i\omega}(z)$, which play important roles below in B.2.2 and in 5.2 in the main text.

Formula (I):   The first formula is

$$\int_0^\infty \frac{dz}{z} K_{i\omega}(z) e^{-iz \sinh t - \epsilon z} = \frac{\pi}{2\omega \sinh \pi\omega} \left(e^{i\omega t} e^{-\pi\omega/2} + e^{-i\omega t} e^{\pi\omega/2} \right), \tag{B.18}$$

where ϵ is an infinitesimal positive parameter, needed to make the integral convergent. To prove this formula, we start with the formula 6.795-1 of [45], which can be expressed as

$$\frac{\pi}{2} e^{-z \cosh \tau} = \int_0^\infty d\omega \cos(\omega\tau) K_{i\omega}(z), \quad |\text{Im}\,\tau| < \frac{\pi}{2}, \quad z > 0. \tag{B.19}$$

By extending the region of ω to $[-\infty, \infty]$ for convenienece[22], the integral on the RHS can be rewritten as

$$\text{RHS} = \frac{1}{2} \int_{-\infty}^\infty e^{i\omega'\tau} K_{i\omega'}(z) d\omega'. \tag{B.20}$$

We now act $\int_0^\infty (dz/z) K_{i\omega}(z)$ on this expression, with ω non-negative. Then using the orthogonality relation (A.1) for $K_{i\omega}(z)$, we get

$$\frac{1}{2} \int_{-\infty}^\infty d\omega' e^{i\omega'\tau} \int_0^\infty \frac{dz}{z} K_{i\omega}(z) K_{i\omega'}(z) = \frac{1}{2} \int_{-\infty}^\infty d\omega' e^{i\omega'\tau} \frac{1}{\mu(\omega)} \left(\delta(\omega' - \omega) + \delta(\omega' + \omega) \right)$$

$$= \frac{1}{2\mu(\omega)} \left(e^{i\omega\tau} + e^{-i\omega\tau} \right) = \frac{1}{\mu(\omega)} \cos \omega\tau, \tag{B.21}$$

where $\mu(\omega)$ is as given in (A.2). Performing the same integral for the LHS as well, (B.19) becomes

$$\int_0^\infty \frac{dz}{z} K_{i\omega}(z) e^{-z \cosh \tau} = \frac{2}{\pi\mu(\omega)} \cos \omega\tau. \tag{B.22}$$

We now make a substitution

$$\tau = t + ((\pi/2) - \epsilon)i, \tag{B.23}$$

where ϵ is an infinitesimal positive quantity. This is legitimate since $\text{Im}\,\tau$ satisfies the condition for the formula (B.19) to be valid. Then by a simple calculation we get $\cosh \tau = \cosh\left(t + \frac{\pi}{2}i - i\epsilon\right) = i\sinh t + \epsilon$, where we reexpressed a positive infinitesimal quantity

---

[22]This is purely as a mathematical equality. The physical energy ω is of course non-negative.

$\epsilon \cosh t$ as ϵ. Substituting (B.23) into the RHS of (B.22) and expanding $\cos \omega \tau$, we obtain the formula (i).

<u>Formula (II)</u> The second formula is

$$A_1(c, k) \equiv \int_0^\infty \frac{dz}{z} K_{i\omega}(kz) e^{-icz - \epsilon z}$$

$$= \frac{\pi}{2\omega \sinh \pi \omega} \left\{ e^{\pi \omega/2} \exp\left(-i\omega \ln\left(\frac{1}{k}\left(c + \sqrt{c^2 + k^2}\right)\right)\right) \right.$$

$$\left. + e^{-\pi \omega/2} \exp\left(i\omega \ln\left(\frac{1}{k}\left(c + \sqrt{c^2 + k^2}\right)\right)\right) \right\}, \qquad (\text{B.24})$$

where c is real and k is real positive. To prove this formula, we first rescale $z \to kz$ in formula (I) and then set $c = k \sinh t$. Solving e^t in terms of c and substituting into the RHS of formula (I), we obtain the integral above.

<u>Formula (III)</u> Finally, a formula similar to (II) we need is

$$A_2(c, k) \equiv \int_0^\infty dz\, K_{i\omega}(kz) e^{-icz}$$

$$= \frac{\pi}{2 \sinh \pi \omega} \frac{1}{\sqrt{c^2 + k^2}} \left\{ e^{\pi \omega/2} \exp\left(-i\omega \ln\left(\frac{1}{k}\left(c + \sqrt{c^2 + k^2}\right)\right)\right) \right.$$

$$\left. - e^{-\pi \omega/2} \exp\left(i\omega \ln\left(\frac{1}{k}\left(c + \sqrt{c^2 + k^2}\right)\right)\right) \right\}. \qquad (\text{B.25})$$

This formula is obtained simply from $A_1(c, k)$ as $A_2(c, k) = i(\partial A_1(c, k)/\partial c)$.

B.2.2 $a_{k\omega}^R$ in terms of $a_{kp^1}^M$

The free scalar field in the right Rindler wedge can be expanded as in (B.8). On the other hand, in this region we should be able to express ϕ^R in terms of ϕ^M and hence $a_{\omega k}^R$ in terms of the Minkowski modes $a_{kp^1}^M$. In the Minkowski spacetime the scalar fields can be written in terms of the coordinate of W_R as

$$\phi^M(z_R, t_R, x) = \int \frac{dp^1}{\sqrt{2\pi}\sqrt{2E_{k',p^1}}} \int \frac{d^2k'}{2\pi} e^{ik'x + ip^1 x^1 - iE_{k',p^1} t_M} a_{k'p^1}^M + \text{h.c.}$$

$$= \int \frac{dp^1}{\sqrt{2\pi}\sqrt{2E_{k',p^1}}} \int \frac{d^2k'}{2\pi} e^{ik'x + ip^1 z_R \cosh t_R - iE_{k',p^1} z_R \sinh t_R} a_{k'p^1}^M + \text{h.c.}, \qquad (\text{B.26})$$

where in the second line we substituted $t_M = z_R \sinh t_R, x^1 = z_R \cosh t_R$. Thus using the Klein-Gordon inner product we can extract the annihilation operators in the right Rindler

coordinate from the expression of the scalar field in the Minkowski spacetime as

$$
\begin{aligned}
a^R_{k\omega} &= (f^R_{k\omega}, \phi^M)^R_{\text{KG}} \\
&= i \int_0^\infty \frac{dz}{z} \int d^2x (f^{R*}_{k\omega} \overleftrightarrow{\partial_{t_R}} \phi^M) \\
&= i \int_0^\infty \frac{dz_R}{z_R} \int d^2x \int \frac{dp^1}{\sqrt{4\pi E_{k'p^1}}} \int \frac{d^2k'}{2\pi} N^R_\omega K^*_{i\omega}(|k|z_R) \\
&\quad \times \left(e^{-i(kx-\omega t_R)} \overleftrightarrow{\partial_{t_R}} [e^{ik'x+ip^1 z_R \cosh t_R - iE_{k'p^1} z_R \sinh t_R} a^M_{k'p^1} + \text{h.c.}] \right) \\
&= i \int \frac{2\pi dp^1 N^R_\omega}{\sqrt{4\pi E_{k'p^1}}} e^{i\omega t_R} \left(\overleftrightarrow{\partial_{t_R}} \int_0^\infty \frac{dz_R}{z_R} K^*_{i\omega}(|k|z_R) e^{-ip^1 z_R \cosh t_R - iE_{kp^1} z_R \sinh t_R} a^M_{kp^1} \right. \\
&\quad \left. + \overleftrightarrow{\partial_{t_R}} \int_0^\infty \frac{dz_R}{z_R} K_{i\omega}(|k|z_R) e^{ip^1 z_R \cosh t_R + iE_{-kp^1} z_R \sinh t_R} a^{M\dagger}_{-kp^1} \right) .
\end{aligned}
$$

Let us now use a convenient parametrization $E_{kp^1} = |k| \cosh \rho, p^1 = -|k| \sinh \rho$, such that $E^2_{kp^1} = (p^1)^2 + k^2$ is realized. Then the expression above can be written as

$$
a^R_{k\omega} = i \int \frac{2\pi dp^1}{\sqrt{2\pi}\sqrt{2E_{k'p^1}}} N^R_\omega e^{i\omega t_R} \overleftrightarrow{\partial_{t_R}} \int_0^\infty \frac{dz_R}{z_R} K_{i\omega}(|k|z_R) \left[e^{-i|k|z_R \sinh(t_R+\rho)} a^M_{kp^1} + e^{i|k|z_R \sinh(t_R+\rho)} a^{M\dagger}_{-kp^1} \right],
$$

where we used the property $K_{i\omega}(z) = K_{-i\omega}(z)$.

Now by using the formula (I) given in (B.18), we can perform the integral over z_R and get

$$
\begin{aligned}
a^R_{k\omega} &= i \int \frac{2\pi dp^1}{\sqrt{2\pi}\sqrt{2E_{kp^1}}} N^R_\omega \frac{\pi}{2\omega \sinh \pi\omega} e^{i\omega t_R} \overleftrightarrow{\partial_{t_R}} \left[\left(e^{-i\omega(t_R+\rho)} e^{\pi\omega/2} + e^{i\omega(t_R+\rho)} e^{-\pi\omega/2} \right) a^M_{kp^1} \right. \\
&\quad \left. + \left(e^{-i\omega(t_R+\rho)} e^{-\pi\omega/2} + e^{i\omega(t_R+\rho)} e^{\pi\omega/2} \right) a^{M\dagger}_{-kp^1} \right] \\
&= \int \frac{dp^1}{\sqrt{2\pi}\sqrt{2E_{kp^1}}} \frac{1}{\sqrt{\sinh \pi\omega}} \left(\frac{E_{kp^1} - p^1}{E_{kp^1} + p^1} \right)^{\frac{-i\omega}{2}} \left[e^{\pi\omega/2} a^M_{kp^1} + e^{-\pi\omega/2} a^{M\dagger}_{-kp^1} \right] . \qquad \text{(B.27)}
\end{aligned}
$$

This is the formula quoted in (2.47). Taking the hermitian conjugation we obtain the creation operator

$$
a^{R\dagger}_{k\omega} = \int \frac{dp^1}{\sqrt{2\pi}\sqrt{2E_{kp^1}}} \frac{1}{\sqrt{\sinh \pi\omega}} \left(\frac{E_{kp^1} - p^1}{E_{kp^1} + p^1} \right)^{\frac{i\omega}{2}} \left[e^{\pi\omega/2} a^{M\dagger}_{kp^1} + e^{-\pi\omega/2} a^M_{-kp^1} \right] . \qquad \text{(B.28)}
$$

B.2.3 $a^F_{k\omega}$ in terms of $a^M_{kp^1}$

As in W_R the free scalar field in W_F frame should be describable in terms of the Minkowski modes. It is expanded as in (2.39) in terms of the Hankel functions $H^{(2)}_{i\omega}(|k|z_F)$, which is recalled in (B.16) for convenience. If we write such a field in the Minkowski

spacetime in terms of the coordinates of W_F, using the relation $t_M = z_F \cosh t_F, x^1 = z_F \sinh t_F$, it reads

$$\phi^M(z_F, t_F, x) = \int \frac{dp^1}{\sqrt{2\pi}\sqrt{2E_{k'p^1}}} \int \frac{d^2k'}{2\pi} e^{ik'x_F + ip^1 z \sinh t_R - iE_{k'p^1} z \cosh t_F} a^M_{k'p^1} + \text{h.c.} . \quad (B.29)$$

Using the Klein-Gordon inner product, we can extract $a^R_{k\omega}$ from the Minkowski field $\phi^M(z_F, t_F, x)$ as

$$a^F_{k\omega} = (f^F_{k\omega}, \phi^M)^F_{\text{KG}} = i \int_{-\infty}^{\infty} z dt_F dx^2 (f^{F*}_{k\omega} \overset{\leftrightarrow}{\partial_z} \phi^M)$$

$$= i \int_{-\infty}^{\infty} z_F dt_F dx^2 \int \frac{dp^1}{\sqrt{2\pi}\sqrt{2E_{k'p^1}}} \int \frac{d^2k'}{2\pi} N^F_\omega e^{-i(kx_F - \omega t_F)}$$

$$\times \left(H^{(2)*}_{i\omega}(|k|z_F) \overset{\leftrightarrow}{\partial_{z_F}} [e^{ik'x_F + ip^1 z_F \sinh t_F - iE_{k'p^1} z_F \cosh t_F} a^M_{k'p^1} + \text{h.c.}] \right)$$

$$= i \int_{-\infty}^{\infty} z_F dt_F \int \frac{dp^1}{\sqrt{2\pi}\sqrt{2E_{kp^1}}} \frac{e^{\pi\omega/2}}{2\sqrt{2}} \left(H^{(2)*}_{i\omega}(|k|z) \overset{\leftrightarrow}{\partial_{z_F}} [e^{ip^1 z_F \sinh t_F - iE_{kp^1} z_F \cosh t_F} e^{i\omega t_F} a^M_{kp^1} \right.$$

$$\left. + e^{-ip^1 z_F \sinh t_F + iE_{-kp^1} z_F \cosh t_F} e^{i\omega t_F} a^{M\dagger}_{-kp^1}] \right) \quad (B.30)$$

We now use the following integral representations [46]

$$\text{(i)} \left(\frac{\alpha + \beta}{\alpha - \beta} \right)^{\nu/2} H^{(1)}_\nu(\sqrt{\alpha^2 - \beta^2}) = \frac{e^{-\nu\pi i/2}}{\pi i} \int_{-\infty}^{\infty} e^{i\alpha \cosh \tau + i\beta \sinh \tau - \nu\tau} d\tau, \quad \text{Im}(\alpha \pm \beta) > 0,$$

$$\text{(ii)} \left(\frac{\alpha + \beta}{\alpha - \beta} \right)^{\nu/2} H^{(2)}_\nu(\sqrt{\alpha^2 - \beta^2}) = -\frac{e^{\nu\pi i/2}}{\pi i} \int_{-\infty}^{\infty} e^{-i\alpha \cosh \tau - i\beta \sinh \tau - \nu\tau} d\tau, \quad \text{Im}(\alpha \pm \beta) < 0.$$

$$(B.31)$$

Note that the formula (i) can be obtained by analytic continuation $\alpha \to e^{i\pi}\alpha, \beta \to e^{i\pi}\beta$ from the formula (ii).

For the part of (B.30) containing $a^M_{kp^1}$, namely

$$H^{(2)*}_{i\omega}(|k|z) \overset{\leftrightarrow}{\partial_z} \int_{-\infty}^{\infty} dt_F e^{ip^1 z_F \sinh t_F - iE_{k'p^1} z_F \cosh t_F} e^{i\omega t_F} a^M_{k'p^1}, \quad (B.32)$$

we can use the formula (ii). On the other hand, for the part containing $a^{M\dagger}_{kp^1}$, i.e.

$$H^{(2)*}_{i\omega}(|k|z) \overset{\leftrightarrow}{\partial_z} \int_{-\infty}^{\infty} dt_F e^{-ip^1 z \sinh t_F + iE_{k'p^1} z \cosh t_F} e^{i\omega t_F} a^{M\dagger}_{k'p^1}, \quad (B.33)$$

it is convenient to use the formula (i). In this way, we can compute (B.30) as

$$a^F_{\omega,k} = -z_F \int \frac{\pi dp^1}{\sqrt{2\pi}\sqrt{2E_{kp^1}}} \frac{e^{\pi\omega/2}}{2\sqrt{2}} \nu \left(H^{(2)*}_{i\omega}(|k|z) \overset{\leftrightarrow}{\partial_z} \left[H^{(2)}_{i\omega}(|k|z) e^{-\pi\omega/2} \left(\frac{E_{kp^1} - p}{E_{kp^1} + p} \right)^{-\frac{i\omega}{2}} a^M_{kp^1} \right.\right.$$

$$\left.\left. - H^{(1)}_{i\omega}(|k|z) e^{\pi\omega/2} \left(\frac{E_{-kp^1} - p}{E_{-kp^1} + p} \right)^{\frac{i\omega}{2}} a^{M\dagger}_{-kp^1} \right] \right)$$

$$= i \int \frac{dp^1}{\sqrt{2\pi E_{-kp^1}}} \left(\frac{E_{kp^1} - p^1}{E_{kp^1} + p^1} \right)^{-\frac{i\omega}{2}} a^M_{kp^1}. \quad (B.34)$$

In the last step, we used the identity (B.14).

Together with the similar result for $a_{k\omega}^{F\dagger}$, we can summarize the results as

$$a_{\omega,k}^{F} = i \int \frac{dp^1}{\sqrt{2\pi E_{kp^1}}} \left(\frac{E_{kp^1} - p^1}{E_{kp^1} + p^1} \right)^{-\frac{i\omega}{2}} a_{kp^1}^{M} ,$$

$$a_{\omega,k}^{F\dagger} = -i \int \frac{dp^1}{\sqrt{2\pi E_{kp^1}}} \left(\frac{E_{kp^1} - p^1}{E_{kp^1} + p^1} \right)^{\frac{i\omega}{2}} a_{kp^1}^{M\dagger} . \tag{B.35}$$

This is the relation quoted in (2.54) in the main text and its hermitian conjugate. As shown in (2.55), in terms of the rapidity variable u defined in (2.48), these relations can be interpreted as the Fourier transforms and then it is practically trivial to check the desired commutation relations

$$[a_{k\omega}^{F}, a_{k'\omega'}^{F\dagger}] = \delta(k - k')\delta(\omega - \omega') , \qquad \text{rest} = 0 . \tag{B.36}$$

B.3 Sketch of the proof that $\phi^M(t_M, x^1, x)$ depends only on the modes of W_L (W_R) for $x^1 < 0$ ($x^1 > 0$)

In this appendix, we give a sketch of the proof that the scalar field in the Minkowski space $\phi^M(t_M, x^1, x)$, when expressed in terms of the oscillators of the Rindler wedge W_R and those of W_L, receive only the contribution of the former(resp. the latter) in the region W_R (resp. W_L).

As in (2.20) in the main text, $\phi^M(t_M, x^1, x)$ is expanded in the plane wave basis as

$$\phi^M(t_M, x^1, x) = \int_{-\infty}^{\infty} \frac{dp^1}{\sqrt{2\pi}\sqrt{2E_{kp^1}}} \int \frac{d^2k}{2\pi} e^{ikx + ip^1x^1 - iE_{kp^1}t_M} a_{kp^1}^{M} + \text{h.c.} \tag{B.37}$$

Now substitute the expression of $a_{kp^1}^{M} = a_{ku}^{M}/\sqrt{E_{kp^1}}$ in terms of $a_{k\omega}^{F}$ given in (2.56) and further use the expressions of $a_{k\omega}^{F}$ and $a_{k,-\omega}^{F}$ in terms of $a_{k\omega}^{R}$ and $a_{k\omega}^{L}$ given in (2.83) and (2.84). This gives ϕ^M in terms of the modes of W_R and W_L. After a simple rearrangement we obtain

$$\phi^M(t_M, x^1, x) = \int \frac{d^2k}{2\pi\sqrt{4\pi}} e^{ikx} \int_0^{\infty} \frac{d\omega}{\sqrt{2\pi}\sqrt{2\sinh\pi\omega}}$$
$$\cdot \left(-I(\omega) \left[e^{\pi\omega/2} a_{k\omega}^{R} - e^{-\pi\omega/2} a_{k\omega}^{L\dagger} \right] + I(-\omega) \left[e^{-\pi\omega/2} a_{k\omega}^{R\dagger} - e^{\pi\omega/2} a_{k\omega}^{L} \right] \right) + \text{h.c.} , \tag{B.38}$$

where

$$I(\omega) \equiv \int_{-\infty}^{\infty} du \, e^{i|k||x^1|\sinh u - i|k|t_M \cosh u - i\omega u} . \tag{B.39}$$

We must study the conditions under which this integral exists. First, for $u \to \infty$, the dominant part of the exponent is $i\frac{|k|}{2}e^u(x^1 - t_M)$. Thus for the integral to converge in

this region, we need the condition $\text{Im}(x^1 + t_M) < 0$. On the other hand for $u \to -\infty$, the dominant part of the exponent is $-i\frac{|k|}{2}(x^1 + t_M)$ and for the convergence we need $\text{Im}(x^1 + t_M) < 0$. These two conditions can be met simultaneously if we make the shift

$$t_M \longrightarrow t_M - i\epsilon, \qquad \epsilon > 0. \tag{B.40}$$

Then, we can make use of the formula 10.9.16 of [46] and get

$$I(\omega) = -i\pi e^{\pi\omega/2} \left(\frac{t_M - x^1 - i\epsilon}{t_M + x^1 - i\epsilon} \right)^{i\omega/2} H_{i\omega}^{(2)}(((t_M - i\epsilon)^2 - (x^1)^2)^{1/2}). \tag{B.41}$$

To express $\phi^M(t_M, x^1, x)$ in (B.38) it is clear that in addition to $I(\omega)$ we need the integrals $I(-\omega)$, $I(\omega)^*$ and $I(-\omega)^*$. To obtain them from $I(\omega)$, we need to make use of the well-known relations among the Hankel functions (see for example 10.46 and 10.11.9 of [46])

$$H_{-i\omega}^{(1)}(z) = e^{-\pi\omega} H_{i\omega}^{(1)}(z), \qquad H_{-i\omega}^{(2)}(z) = e^{\pi\omega} H_{i\omega}^{(2)}(z), \tag{B.42}$$

$$H_{i\omega}^{(1)}(z)^* = H_{-i\omega}^{(2)}(z^*) = e^{\pi\omega} H_{i\omega}^{(2)}(z^*), \tag{B.43}$$

$$H_{i\omega}^{(2)}(z)^* = H_{-i\omega}^{(1)}(z^*) = e^{-\pi\omega} H_{i\omega}^{(1)}(z^*). \tag{B.44}$$

We then get

$$I(\omega) = -i\pi e^{\pi\omega/2} \left(\frac{t_M - x^1 - i\epsilon}{t_M + x^1 - i\epsilon} \right)^{i\omega/2} H_{i\omega}^{(2)}(((t_M - i\epsilon)^2 - (x^1)^2)^{1/2}), \tag{B.45}$$

$$I(-\omega) = -i\pi e^{\pi\omega/2} \left(\frac{t_M - x^1 - i\epsilon}{t_M + x^1 - i\epsilon} \right)^{-i\omega/2} H_{i\omega}^{(2)}(((t_M - i\epsilon)^2 - (x^1)^2)^{1/2}), \tag{B.46}$$

$$I(\omega)^* = i\pi e^{-\pi\omega/2} \left(\frac{t_M - x^1 + i\epsilon}{t_M + x^1 + i\epsilon} \right)^{-i\omega/2} H_{i\omega}^{(1)}(((t_M + i\epsilon)^2 - (x^1)^2)^{1/2}), \tag{B.47}$$

$$I(-\omega)^* = i\pi e^{-\pi\omega/2} \left(\frac{t_M - x^1 + i\epsilon}{t_M + x^1 + i\epsilon} \right)^{i\omega/2} H_{i\omega}^{(1)}(((t_M + i\epsilon)^2 - (x^1)^2)^{1/2}). \tag{B.48}$$

Now rather than displaying the complete expression for $\phi^M(t_M, x^1, x)$, it should suffice to demonstrate that the coefficient of $a_{k\omega}^R$ vanishes in W_L region, as the rest of the calculations are entirely similar.

Let us note that $a_{k\omega}^R$ appears in two places, namely $a_{k\omega}^F$ part of $a_{kp^1}^M$ and $a_{-\omega k}^{F\dagger}$ part of $a_{kp^1}^{M\dagger}$. The total contribution for the coefficient of $a_{k\omega}^R$ from these sources is proportional to $-I(\omega)e^{\pi\omega/2} + I(-\omega)^* e^{\pi\omega/2}$.

Consider now the region W_L, where $t_M + x^1 < 0$, $t_M - x^1 > 0$ and of course $x_1 < 0$. Thus, apart from the $\pm i\epsilon$, we have $t_M^2 - (x^1)^2 = -z_L^2 < 0$ and we must choose the square root branch for the quantity $(-z^2)^{1/2}$. (Since $z_L = z_R$, we denote it by z for simplicity hereafter.) As a concrete choice, let us take the branch cut to be along $[-\infty, 0]$ in the z plane. This means that $(-z^2 \pm i\delta)^{1/2} = \pm iz$ for small positive δ.

First consider the region of W_L where $t_M > 0$. Then, we have $\delta = \epsilon t_M$ and in the expressions of $-I(\omega)$ and $I(-\omega)^*$, we have, respectively, $H^{(2)}_{i\omega}(-iz)$ and $H^{(1)}_{i\omega}(iz)$. In this case, from the formula 10.11.5 of [46], we have

$$H^{(2)}_{i\omega}(e^{-i\pi}iz) = -e^{-\pi\omega}H^{(1)}_{i\omega}(iz) \,. \tag{B.49}$$

Using this relation, it is easy to see that $-I(\omega) + I(-\omega)^* = 0$ and the coefficient of $a^R_{i\omega}$ vanishes in W_L , as desired.

Next, consider the region in W_L where $t_M < 0$. Then, we have instead $H^{(2)}_{i\omega}(iz)$ for $-I(\omega)$ and $H^{(1)}_{i\omega}(-iz)$ for $I(-\omega)^*$. Then, again from 10.11.5 of [46], we have

$$H^{(1)}_{i\omega}(e^{i\pi}iz) = -e^{\pi\omega}H^{(2)}_{i\omega}(iz) \,, \tag{B.50}$$

and $-I(\omega)$ and $I(-\omega)^*$ cancel with each other in this case as well.

Combining, we have shown that $a^R_{\omega k}$ does not contribute in the expansion in the region W_L .

C Poincaré algebra for the various observers

C.1 Proof of the Poincaré algebra in W_F frame

In this appendix, we shall demonstrate that the generators M^F_{01}, H^F, and P^F_1 constructed in (2.97), (2.101) and (2.102) form the Poincaré algebra.

First, consider the commutator $[H^F, M^F_{01}]$. This can be computed as

$$
\begin{aligned}
[H^F, M^F_{01}] &= -\int d^2k' \, |k'| \int d^{d-2}k \int d\omega' d\omega [a^{F\dagger}_{k\omega'} \cos\left(\frac{d}{d\omega'}\right) a^F_{k\omega'}, \omega a^{F\dagger}_{k\omega} a^F_{k\omega}] \\
&= -\int d^2k \, |k| \int d\omega \, \omega a^{F\dagger}_{k\omega} \left(\cos\left(\frac{d}{d\omega}\right)(\omega a^F_{k\omega}) - \omega \cos\left(\frac{d}{d\omega}\right) a^F_{k\omega} \right) \\
&= \int d^2k \, |k| \int d\omega a^{F\dagger}_{k\omega} \sin\left(\frac{d}{d\omega}\right) a^F_{k\omega} \\
&= iP^F_1, \tag{C.1}
\end{aligned}
$$

where in the second line we used the simple identity

$$\left(\frac{d}{d\omega}\right)^n (\omega a_\omega) = n \left(\frac{d}{d\omega}\right)^{n-1} a_\omega + \omega \left(\frac{d}{d\omega}\right)^n a_\omega \,. \tag{C.2}$$

In an entirely similar manner, with cos and sin interchanged, $[P^F_1, M^F_{01}] = iH^F$ can be shown.

Finally, the fact that $[H^F, P_1^F]$ vanishes can be checked as

$$
\begin{aligned}
[H^F, P_1^F] &= -i \int d^{d-2}k' \, |k'| \int d^{d-2}k|k| \int d\omega' d\omega [a_{k\omega'}^{F\dagger} \cos\left(\frac{d}{d\omega'}\right) a_{k\omega'}^F, a_{k\omega}^{F\dagger} \sin\left(\frac{d}{d\omega}\right) a_{k\omega}^F] \\
&= -i \int d^{d-2}k \, |k|^2 \int d\omega \omega a_{k\omega}^{F\dagger} \left(\sin\left(\frac{d}{d\omega}\right)\cos\left(\frac{d}{d\omega}\right)a_{k\omega}^F - \cos\left(\frac{d}{d\omega}\right)\sin\left(\frac{d}{d\omega}\right)a_{k\omega}^F\right) \\
&= 0 \, .
\end{aligned} \tag{C.3}
$$

C.2 Poincaré generators for $\mathbf{W_F}$ by the unitary transformation

Recall that the unitary transformation U_F defined by

$$
U_F = e^{-\frac{i\pi}{2}A}, \qquad A = \frac{1}{2}\int_{-\infty}^{\infty} d\omega a_{k\omega}^{F\dagger}(-\frac{d^2}{d\omega^2} + \omega^2 - 1)a_{k\omega}^F \, , \tag{C.4}
$$

converts the mode operator $a_{k\omega}^F$ into $a_{kp^1}^M$ in the manner

$$
U_F a_{k\omega}^F U_F^\dagger = i\sqrt{E_{kp^1}} a_{kp^1}^M \, . \tag{C.5}
$$

As an application of this operation, let us show that it transforms M_{01}^F into M_{01}, namely

$$
U_F \left(\int d^2k \int_{-\infty}^{\infty} d\omega \, \omega a_{k\omega}^{F\dagger} a_{k\omega}^F\right) U_F^\dagger = i \int d^2k \int_{-\infty}^{\infty} dp^1 \, E_{kp^1} \, a_{kp^1}^{M\dagger} \frac{\partial}{\partial p_1} a_{kp^1}^M \, . \tag{C.6}
$$

First, expand the unitary transformation as the sum of multiple commutators in the usual way:

$$
U_F M_{01}^F U_F^\dagger = M_{01}^F - \frac{i\pi}{2}[A, M_{01}^F] + \frac{1}{2!}\left(-\frac{i\pi}{2}\right)^2 [A, [A, M_{01}^F]] + \cdots . \tag{C.7}
$$

The single commutator can be computed as

$$
\begin{aligned}
[A, M_{01}^F] &= [A, \int d^2k \int_{-\infty}^{\infty} d\omega \omega a_{k\omega}^{F\dagger} a_{k\omega}^F] \\
&= \frac{1}{2}\int d^2k \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' [a_{k\omega'}^{F\dagger}\left(-\frac{d^2}{d\omega'^2}\right)a_{k\omega'}^F, \omega a_{k\omega}^{F\dagger} a_{k\omega}^F] \\
&= -\frac{1}{2}\int d^2k \int_{-\infty}^{\infty} d\omega \left[a_{k\omega}^{F\dagger}\left(\frac{d^2}{d\omega^2}\right)\omega a_{k\omega}^F - \omega a_{k\omega}^{F\dagger}\frac{d^2}{d\omega^2}a_{k\omega}^F\right] \\
&= -\int d^2k \int_{-\infty}^{\infty} d\omega a_{k\omega}^{F\dagger}\frac{d}{d\omega}a_{k\omega}^F \, . \tag{C.8}
\end{aligned}
$$

Based on this result, the double commutator is calculated as

$$
\begin{aligned}
[A, [A, M_{01}^F]] &= [A, -\int d^2k \int_{-\infty}^{\infty} d\omega a_{k\omega}^{F\dagger}\frac{d}{d\omega}a_{k\omega}^F] \\
&= -\frac{1}{2}\int d^2k \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega'(\omega'^2 - 1)[a_{k\omega'}^{F\dagger}a_{k\omega'}^F, a_{k\omega}^{F\dagger}\frac{d}{d\omega}a_{k\omega}^F] \\
&= -\frac{1}{2}\int d^2k \int_{-\infty}^{\infty} d\omega[(\omega^2 - 1)a_{k\omega}^{F\dagger}\frac{d}{d\omega}a_{k\omega}^F - a_{k\omega}^{F\dagger}\frac{d}{d\omega}(\omega^2 - 1)a_{k\omega}^F] \\
&= \int d^2k \int_{-\infty}^{\infty} d\omega \, \omega a_{k\omega}^{F\dagger}a_{k\omega}^F \, . \tag{C.9}
\end{aligned}
$$

Since this is of the original form of M_{01}^F, we see that the rest of the multiple commutators produce $\int d^2k \int_{-\infty}^{\infty} d\omega\, a_{k\omega}^{F\dagger} \frac{d}{d\omega} a_{k\omega}^F$ and $\int d^2k \int_{-\infty}^{\infty} d\omega\, \omega a_{k\omega}^{F\dagger} a_{k\omega}^F$ alternately. The coefficients can be easily found in such a way that the series sum up to

$$U_F M_{01}^F U_F^\dagger = \cos\frac{\pi}{2} \int d^2k \int_{-\infty}^{\infty} d\omega\, \omega a_{k\omega}^{F\dagger} a_{k\omega}^F + i\sin\frac{\pi}{2} \int d^2k \int_{-\infty}^{\infty} d\omega\, a_{k\omega}^{F\dagger} \frac{d}{d\omega} a_{k\omega}^F$$
$$= i \int d^2k \int_{-\infty}^{\infty} d\omega\, a_{k\omega}^{F\dagger} \frac{d}{d\omega} a_{k\omega}^F \,. \tag{C.10}$$

Making the replacements $\omega \to p^1$ and $a_{k\omega}^F \to \sqrt{E_{kp^1}}\, a_{kp^1}^M$, we obtain the desired result

$$U_F M_{01}^F U_F^\dagger = i \int d^2k \int dp^1\, E_{kp^1}\, a_{kp^1}^{M\dagger} \frac{\partial}{\partial p_1} a_{kp_1}^M = M_{01}\,. \tag{C.11}$$

D Quantization in different Lorentz frames with an almost light-like boundary condition

Here we supply some details of the quantization in different Lorentz frames with a slightly timelike boundary condition discussed in Sec. 4.2.4.

What we shall describe is the computations of the two terms (4.32) and (4.33) which constitute the commutator $\left[\tilde{\pi}(\tilde{t}, \tilde{x}^1, \Omega), \phi(\tilde{t}, \tilde{y}^1, \Omega')\right]$ given in (4.31). For the convenience of the reader let us display them again:

$$C_1 = -\gamma \widetilde{\mathcal{N}}^2 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_0^{\infty} \frac{d\hat{p}^1}{4\pi} Y_{lm}(\Omega) Y_{lm}^*(\Omega') \left(e^{i\beta \hat{E}_{k_l\hat{p}^1}(\hat{x}^1 - \hat{y}^1)} + \text{h.c.}\right) \sin\hat{p}^1\hat{x}^1\,\sin\hat{p}^1\hat{y}^1\,,$$
$$\tag{D.1}$$

$$C_2 = -i\gamma\beta\widetilde{\mathcal{N}}^2 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_0^{\infty} \frac{d\hat{p}^1}{4\pi\hat{E}_{k_l\hat{p}^1}} Y_{lm}(\Omega) Y_{lm}^*(\Omega') \left(e^{i\beta\hat{E}_{k_l\hat{p}^1}(\hat{x}^1-\hat{y}^1)} - \text{h.c.}\right) \cos\hat{p}^1\hat{x}^1\,\sin\hat{p}^1\hat{y}^1\,.$$
$$\tag{D.2}$$

First the sum over m can be performed by the addition theorem for Y_{lm} as already described in (4.34). Next, we perform the integral over $\hat{p}^1$. Although the energy dependence in the exponent does not disappear at equal $\tilde{t}$, in contrast to the case for the frame $(\hat{t}, \hat{x}^1)$, such an integral can be performed, after expressing the product of trigonometric functions into a sum like $\sin\hat{p}^1\hat{x}^1 \sin\hat{p}^1\hat{y}^1 = \frac{1}{2}\left(\cos\hat{p}^1(\hat{x}^1 - \hat{y}^1) - \cos\hat{p}^1(\hat{x}^1 + \hat{y}^1)\right)$. The relevant formulas were given in (4.35) and (4.36), with appropriate regularizations (4.37) and (4.38) for convergence.

Then, the result for $C_1 + C_2$ takes the form

$$C_1 + C_2 = -i\frac{\beta\gamma\widetilde{\mathcal{N}}^2}{8\pi} \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\hat{n} \cdot \hat{n}')$$

$$\times \left\{ -\frac{a_- + i\eta}{\sqrt{a_-^2 + b_-^2}} K_1(k_l\sqrt{a_-^2 + b_-^2}) + \frac{a_- - i\eta}{\sqrt{a_-^2 + b_+^2}} K_1(k_l\sqrt{a_-^2 + b_+^2}) \right.$$

$$+ \frac{a_- + i\eta}{\sqrt{a_+^2 + b_-^2}} K_1(k_l\sqrt{a_+^2 + b_-^2}) - \frac{a_- - i\eta}{\sqrt{a_+^2 + b_+^2}} K_1(k_l\sqrt{a_+^2 + b_+^2})$$

$$+ \frac{a_+}{\sqrt{a_+^2 + b_-^2}} K_1(k_l\sqrt{a_+^2 + b_-^2}) - \frac{a_+}{\sqrt{a_+^2 + b_+^2}} K_1(k_l\sqrt{a_+^2 + b_+^2})$$

$$\left. - \frac{a_-}{\sqrt{a_-^2 + b_-^2}} K_1(k_l\sqrt{a_-^2 + b_-^2}) + \frac{a_-}{\sqrt{a_-^2 + b_+^2}} K_1(k_l\sqrt{a_-^2 + b_+^2}) \right\}, \qquad (D.3)$$

where

$$a_\pm \equiv \hat{x}^1 \pm \hat{y}^1, \qquad b_\pm \equiv \pm i\beta(\hat{x}^1 - \hat{y}^1 \mp i\eta). \qquad (D.4)$$

Consider first the four terms in the third and the fourth lines, which contain a_+^2 in the square roots of the denominator and in the argument of K_1 functions. Since $\hat{x}^1$ and $\hat{y}^1$ are positive, a_+ is positive and generically finite. Therefore, we can ignore η for these terms. Then, $b_+^2 = b_-^2$ and hence the two terms in the third line cancel and similarly the the two terms in the fourth line cancel. Therefore these four terms actually do not contribute and we can simplify $C_1 + C_2$ to

$$C_1 + C_2 = -i\frac{\beta\gamma\widetilde{\mathcal{N}}^2}{8\pi} \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\hat{n} \cdot \hat{n}')$$

$$\times \left\{ -\frac{a_- + i\eta}{\sqrt{a_-^2 + b_-^2}} K_1(k_l\sqrt{a_-^2 + b_-^2}) + \frac{a_- - i\eta}{\sqrt{a_-^2 + b_+^2}} K_1(k_l\sqrt{a_-^2 + b_+^2}) \right.$$

$$\left. - \frac{a_-}{\sqrt{a_-^2 + b_-^2}} K_1(k_l\sqrt{a_-^2 + b_-^2}) + \frac{a_-}{\sqrt{a_-^2 + b_+^2}} K_1(k_l\sqrt{a_-^2 + b_+^2}) \right\}, \qquad (D.5)$$

To analyze this expression we must distinguish two regions.

(i) If a_-^2 is finite, then we can again ignore η and these four terms cancel in exactly the same fashion.

(ii) Thus non-vanishing result can possibly be obtained if and only if $|a_-| \lesssim \eta$. In such a case, since a_- and $b_\pm$ are of the order η, as long as k_l is not infinite, we can use the approximation $K_1(z) \simeq 1/z$ and hence each term diverges like $1/\eta$.

Combining, this shows that the sum of terms containing K_1 function behaves precisely like $\sim \delta(\hat{x}^1 - \hat{y}^1)$. The rest of the argument is already given in the main text and the commutator $[\tilde{\pi}(\tilde{t}, \tilde{x}^1, \Omega), \phi(\tilde{t}, \tilde{y}^1, \Omega')]$ in the frame of an arbitrary FFO correctly behaves like the product of appropriate δ-functions.

Thus, we can write

$$\left[\tilde{\pi}(\tilde{t}, \tilde{x}^1, \Omega), \phi(\tilde{t}, \tilde{y}^1, \Omega')\right] = C_1 + C_2 = -iF\gamma\delta(\hat{x}^1 - \hat{y}^1)\delta(\cos\theta - \cos\theta')\delta(\varphi - \varphi'), \quad \text{(D.6)}$$

where we used the relation $\delta(\tilde{x}^1 - \tilde{y}^1) = \gamma\delta(\hat{x}^1 - \hat{y}^1)$ valid at equal $\tilde{t}$. F is a constant, which we want to set to unity by adjusting the normalization constant $\widetilde{\mathcal{N}}$. To find such $\widetilde{\mathcal{N}}$, we need to carry out the integral $i\int d\hat{x}^1 d\cos\theta d\varphi (C_1 + C_2)$, perform the sum over l and set the result to 1. This unfortunately is quite difficult and we have not been able to find the form of $\widetilde{\mathcal{N}}$ explicitly.

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