Multi modification method toward solving sign problem

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A possible method to solve the sign problem is developed by modifying the original theory. Considering several modifications of the theory, the observable in the original theory is reconstructed from the identity connecting the observables in the original and modified theories. We demonstrate that our method gives the correct results even if the original theory has the severe sign problem by using a simple 1-dimensional integral.

I. INTRODUCTION

In the many fields of physics, the first-principle calculation plays an important role to analyze the nonperturbative properties of theories. However, when the action S is complex valued, the first-principle calculation is difficult because of the cancellation between the Boltzmann factors e^{-S}. This problem is referred to as the sign problem. Although several ways have been proposed to solve the sign problem, it has not been solved yet. A possible solution for the sign problem is the complex Langevin method. The complex Langevin method is based on the stochastic quantization [1] with complex actions [2, 3, 4, 5] (see [6, 7] for a review). While the original Langevin method with real actions always gives the correct results, the complex Langevin method sometimes fails to reproduce the correct results [8, 9, 10, 11]. The condition of the correctness the complex Langevin method has been discussed [12, 13]. If the distribution of the complex variables in the complex Langevin dynamics overlaps the singularity of the drift term of the complex Langevin equation, it breaks a requirement for the correctness of the complex Langevin method. Recently, it is shown that this problem is one of the reasons for the failure of the complex Langevin method and it is referred to as the singular drift problem [14, 15]. As other reasons, the complex Langevin fails when the distribution of the variables in the complex Langevin dynamics has a tail of slow decay in the complex direction [16] and when the ergodicity of the complex Langevin dynamics is not satisfied [17]; see Ref. [18] a recent review of the complex Langevin method. The applicable scope of the complex Langevin in Quantum Chromodynamics(QCD) at finite density has been investigated [17, 19, 20, 21, 22, 23, 24].

The purpose of this paper is to develop a way to obtain the correct results even when the complex Langevin method fails to reproduce the correct results. Our method proposed in this paper is a improvement of the method proposed in our previous paper [25]. In the previous paper, we proposed a new idea to avoid the sign problem. In this approach, the expectation value of observables in a *bad* model, which suffers from the sign problem is reconstructed by ones in a *good* model which is free from the sign problem through a simple identity. Here, the *good* model is defined by adding an analytic function to the fermionic determinant of the original *bad* model so that the modified model has a desirable property from a view point of some numerical computational schemes to be applied. Thus, we refer to this approach as the modification method. In the previous work however, we implicitly assumed that the reweighting factor involved in the identity can be always computed within appropriate precisions. Obviously, this assumption will not be satisfied when the sign problem is quite severe. Thus, the applicability of the modification method is expected not be different so much from others investigated so far, such as reweighting method [26, 27, 28, 29].

In this paper, we propose a new method and demonstrate that it is applicable without computing the reweighting factor. In Sec. II, we review the modification method proposed in the our previous paper [25] and point out that the reweighting factor appears in the key identity and the method is not applicable when the reweighting factor is small. Then, we improve our method and the actual procedure is explained in the general case. In Sec. III, we apply our method to a simple model, the Gaussian model, and demonstrate that our method reproduces the correct results. Then, we discuss some properties of the our method, such as the advantages, the difference from the reweighting method and the applicable scope. Section IV presents our conclusions.

II. MODIFICATION METHOD

We focus on the class of models whose partition function has a following form:

$$Z_f = \int_D dx f(x) \mathrm{e}^{-S_{\mathrm{q}}(x)}, \tag{1}$$

where $f(x)$ is a complex-valued function defined on $x \in \mathbb{R}$, $S_{\mathrm{q}}(x)$ is a real-valued action and D is an integration domain on a real axis, $D \subset \mathbb{R}$. Since one can generalize our formulation to a higher dimensional theory in a

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straightforward way, we consider 0-dimensional field theory, namely 1-dimensional integral. Typically, this type of the partition function appears by integrating out the fermion sector from the action. For instance, this class of models includes the Thirring model, random matrix models, and QCD. In these cases, $f(x)$ corresponds to the fermion determinant.

By exponentiating $f(x)$ in Eq. 1, we get

$$Z_f = \int_D dx \mathrm{e}^{-(S_\mathrm{q}(x) - \log f(x))}. \tag{2}$$

Clearly seen from this expression, the effective action $S(x) \equiv S_\mathrm{q}(x) - \log f(x)$ is complex unless $f(x)$ always take real and positive values, that is $f(x) \in \mathbb{R}_+$. When $S(x)$ is complex valued function, it causes the sign problem for this class of models.

The expectation value of an observable $\mathcal{O}(x)$ is defined by

$$\langle \mathcal{O} \rangle_f \equiv \frac{1}{Z_f} \int_D dx \mathcal{O}(x) f(x) \mathrm{e}^{-S_\mathrm{q}(x)}. \tag{3}$$

For the later convenience, we introduce the special notation for the case of $f(x) \equiv 1$:

$$Z \equiv Z_1 = \int_D dx \mathrm{e}^{-S_\mathrm{q}(x)}, \quad \langle \mathcal{O} \rangle \equiv \langle \mathcal{O} \rangle_1 . \tag{4}$$

With these definitions, the observable $\mathcal{O}(x)$ obeys

$$\langle f \rangle \langle \mathcal{O} \rangle_f = \langle f \mathcal{O} \rangle . \tag{5}$$

By using the identity, we find the following relation for two arbitrary complex-valued functions $f(x)$ and $g(x)$ and the observable $\mathcal{O}(x)$:

$$\langle f \rangle \langle \mathcal{O} \rangle_f + \langle g \rangle \langle \mathcal{O} \rangle_g = \langle f + g \rangle \langle \mathcal{O} \rangle_{f+g} . \tag{6}$$

If $\langle f \rangle \neq 0$, we obtain

$$\langle O \rangle_f = \langle O \rangle_{f+g} + \left(\langle O \rangle_{f+g} - \langle O \rangle_g \right) \frac{\langle g \rangle}{\langle f \rangle}. \tag{7}$$

This is what we have shown in the previous paper [25]. This identity is useful when the expectation value $\langle O \rangle_f$ is difficult to compute due to the sign problem. If one chooses an appropriate $g(x)$ so that the alternative model Z_g and the *modified* model Z_{f+g} are free from the sign problem, one obtains $\langle O \rangle_f$ through computing $\langle O \rangle_g$ and $\langle O \rangle_{f+g}$. We refer to this technique as the modification method. It is known that this method is applicable to the U(1)-link model [10]. A practical way to find an optimal $g(x)$ is also proposed in the previous work [25].

However, there is a caveat. It is non-trivial whether $\langle O \rangle_f$ can be computed within appropriate precisions due to the existence of the factor $\langle g \rangle / \langle f \rangle$. In fact, this factor is rewritten as

$$\frac{\langle g \rangle}{\langle f \rangle} = \frac{\int dx g(x) \mathrm{e}^{-S\mathrm{q}(x)}}{\int dx f(x) \mathrm{e}^{-S\mathrm{q}(x)}} \tag{8}$$

$$= \frac{\int dx g(x) \mathrm{e}^{-S\mathrm{q}(x)}}{\int dx \left(\frac{f(x)}{g(x)} \right) g(x) \mathrm{e}^{-S\mathrm{q}(x)}} = \left\langle \frac{f}{g} \right\rangle_g^{-1}. \tag{9}$$

The quantity $\langle f/g \rangle_g$ is nothing but the so-called reweighting factor [30]. If the absolute value of the reweighting factor is small, it indicates that the sign problem is severe. Apparently, one can compute $\langle \mathcal{O} \rangle_f$ through Eq. 7 when $g(x)$ is chosen so that the factor $|\langle f/g \rangle_g|$ is sufficiently large. Therefore, it seems that our modification method has the same difficulty as the reweighting method [26].

Nevertheless, we shall point out that our method is different from the reweighting method and the large reweighting factor is not necessary to apply the modification method. To see this, we rewrite Eq. 7 as

$$y = a_g x + b_g. \tag{10}$$

Here, we separate the Eq. 7 into the g-independent parts

$$y = \langle O \rangle_f , \quad x = \frac{1}{\langle f \rangle}, \tag{11}$$

and g-dependent parts

$$a_g = \left(\langle O \rangle_{f+g} - \langle O \rangle_g \right) \langle g \rangle , \tag{12}$$

$$b_g = \langle O \rangle_{f+g} . \tag{13}$$

Suppose that all g-dependent quantities can be computed without the sign problem, and there are several candidates of such functions $g(x)$. In this case, g-independent quantities y and x are obtained as an intersection point of the set of straight lines $\{a_g x + b_g\}$. In the following section, we demonstrate the method discussed in this section, which is referred to as the multi-modification method in this paper, using a simple model.

III. APPLICATION TO THE GAUSSIAN MODEL

In this section, we apply our method to the Gaussian model. To begin with, the sign problem in the model is discussed. Then, we demonstrate how our method works and gives the correct results.

A. Sign problem in the Gaussian model

The Gaussian model is a 0-dimensional field theory, which has the partition function

$$Z_f = \int_{-\infty}^{\infty} dx f(x) e^{-x^2/2}, \quad f(x) = (x + i\alpha)^2 \tag{14}$$

with a positive real parameter $\alpha > 0$. The effective action $S(x) = x^2/2 - \log f(x)$ is complex valued because $f(x)$ is complex valued when $\alpha \neq 0$. Then, this model has the sign problem.

The analytic solutions of the observables in this model can be easily obtained. For example, the observables x^2, x^4, x^6 are obtained as

$$\langle x^2 \rangle_f = \frac{3 - \alpha^2}{1 - \alpha^2} \tag{15}$$

$$\langle x^4 \rangle_f = \frac{15 - 3\alpha^2}{1 - \alpha^2} \tag{16}$$

$$\langle x^6 \rangle_f = \frac{105 - 15\alpha^2}{1 - \alpha^2}. \tag{17}$$

In this study, we adopt the complex Langevin method to numerically calculate the observables in the Gaussian model. The complex Langevin equation of this model is written as

$$\frac{dz}{dt} = D(z) + \eta(t), \tag{18}$$

where z is the complexified variable $x \to z \in \mathbb{C}$ and t is the fictitious time of the complex Langevin dynamics. The First term of the RHS in Eq. (18) is the drift term, which is expressed as

$$D(z) = -\frac{\partial S(z)}{\partial z} = -z + \frac{2}{z + i\alpha} \tag{19}$$

and the second term is the Gaussian noise term, which satisfies

$$\langle \eta(t) \rangle_\eta = 0, \quad \langle \eta(t_1)\eta(t_2) \rangle_\eta = 2\delta_{t_1,t_2}, \tag{20}$$

where $\langle \cdots \rangle_\eta$ denotes the noise average. It was shown that the complex Langevin method in this model gives the correct results when the singular drift problem does not occur [14, 15]. In this model, there are no problems on the slow decay of the distribution in the complex direction and the ergodicity. Thus in this model, we suppose that the condition for the correctness of the complex Langevin method is only the absence of the singular drift problem.

In the Gaussian model, the correctness of the complex Langevin method depends on the parameter α. In Fig. 1, the numerical results of the complex Langevin method for an observable $O(z) = \mathrm{Re}(z^2)$ are shown. When the parameter α is sufficiently large $\alpha \gtrsim 2.7$, the complex Langevin method gives the correct result. This is because there is no singular drift problem when $\alpha \gtrsim 2.7$. To see that, in Fig. 2, the histogram of the absolute value of drift term $\rho(|D(z)|)$ in the complex Langevin dynamics is shown when $\alpha = 2.7$. In the numerical calculation, we use the Euler's method to solve the complex Langevin equation for the total Langevin flow time 10^7 with the discrete time $dt = 10^{-2}$. We take configuration every 1 Langevin time after 10^2 Langevin time for the thermalization. Since the histogram in Fig. 2 exponentially damps when $\alpha = 2.7$, it is found that the singular drift problem does not occur. On the other hand, the complex Langevin method fails to give the correct results in the other region, in particular around $\alpha \simeq 1$. In Fig. 3, the distribution of the drift term with $\alpha = 1.5$ is shown. Unlike the case with $\alpha \gtrsim 2.7$, the distribution with $\alpha = 1.5$ does not exponentially drop and has the long tail to the large $|D|$. Thus, when $\alpha = 1.5$, it seems that the singular drift problem occurs. Such kind of the analysis has been done in the previous study in the Gaussian model [14].

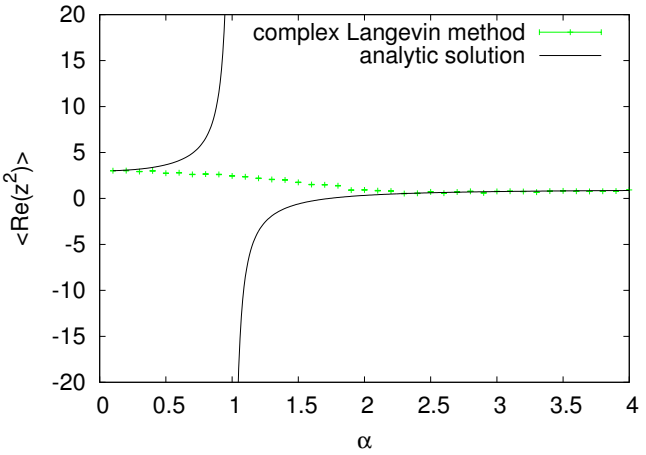

FIG. 1. The observables $\langle \mathrm{Re}(z^2) \rangle$ plotted against the parameter α in the Gaussian model. Solid line denotes the analytical solution and the points denotes the numerical results of the complex Langevin method.

In this paper, in order to reproduce the correct results even in the small-α region by numerical calculation, we consider the modification method. However, it is difficult to calculate observables within sufficient precision only by directly using the modification method in the Gaussian model. To see that, in the following, we consider to directly apply the modification formula (7) to the Gaussian model.

As a modification function g(x) of the theory, we consider a function

$$g(x) = (x + i\beta)^2 \tag{21}$$

with a positive real parameter β. This function has the same form as $f(x)$ in Eq. (14) and different parameter β. In order to apply the modification method, we have to

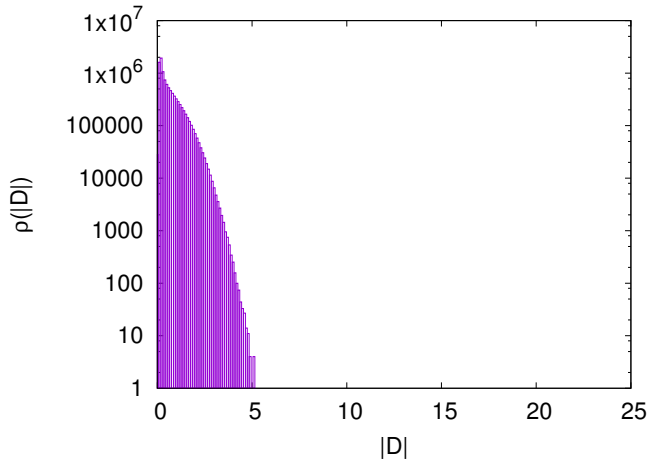

FIG. 2. The histogram for the absolute value of the drift term in the complex Langevin equation of the Gaussian model when $\alpha = 2.7$ in log scale.

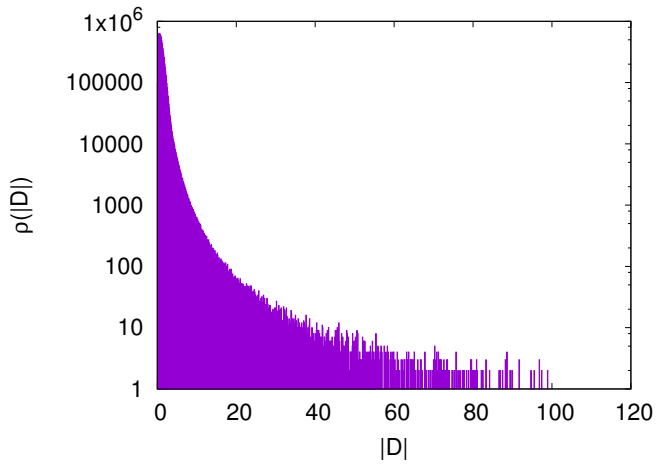

FIG. 3. The histogram for the absolute value of the drift term in the complex Langevin equation of the Gaussian model when $\alpha = 1.5$ in log scale.

choose appropriate function $g(x)$, namely the parameter β here, so that the following quantities can be correctly calculated:

1. The observables $\langle O \rangle_g$ and $\langle O \rangle_{f+g}$
2. The factor $\langle g \rangle / \langle f \rangle$

Concretely, the value of β is constrained in the following way. The observable $\langle O \rangle_g$ in the g-theory should be correctly calculated by the complex Langevin method. Since the modification function $g(x)$ has the same form as the original function $f(x)$, we already know that $\langle O \rangle_g$ is correctly obtained when $\beta > 2.7$. In addition to the observable $\langle O \rangle_g$, the observable $\langle O \rangle_{f+g}$ in the $(f+g)$-theory also should be correctly calculated by the complex Langevin method. We perform the complex Langevin dynamics and investigate the distribution of the drift term in the $(f+g)$-theory for each α and β. For example, in Fig. 4, it is shown that the distribution of the absolute value of the drift term exponentially drops and the singular drift problem does not occur when $\alpha = 1.5$ and $\beta = 3.5$. In our analysis on the distribution of the drift term, we find that there is no singular drift problem in the $(f+g)$-theory for arbitrary $\alpha > 0$ if we choose $g(x)$ with $\beta \geq 3.5$. Thus, the 1st condition of β is $\beta \geq 3.5$.

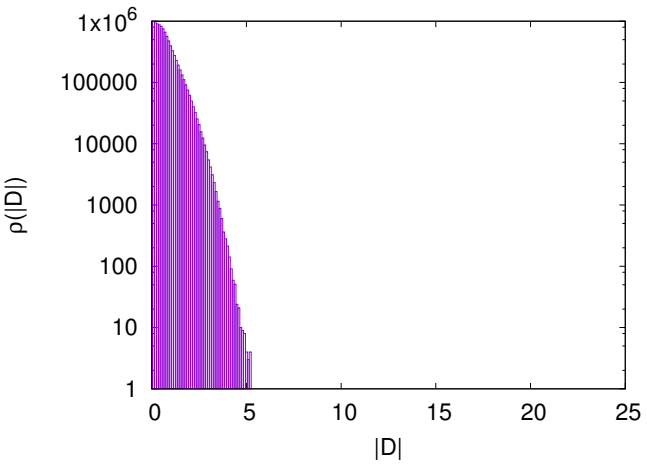

FIG. 4. The histogram for the absolute value of the drift term in the complex Langevin equation of the modified Gaussian model when $\alpha = 1.5$ and $\beta = 3.5$ in log scale.

Next, following the 2nd condition, the factor $\langle g \rangle / \langle f \rangle$ should be correctly calculated. However, this factor is problematic in the modification method. As discussed in (9) in the section II, this factor is rewritten to the inverse of the reweighting factor $\langle f/g \rangle_g^{-1}$. When the sign problem is severe, the absolute value of the reweighting factor $|\langle f/g \rangle_g|$ tends to be small, and it is difficult to calculate the observable $\langle O \rangle_f$ with sufficient precision. Actually, as shown in Fig. 5, the absolute value of the reweighting factor $|\langle f/g \rangle_g|$ is smaller than 1 in $0 < \alpha < 3$ with $3.5 \leq \beta$. In particular, in the region around $\alpha = 1$, where the sign problem is severe, the reweighting factor is almost 0. This happens because of the oscillatory behavior of the quantity $\langle f \rangle$. Thus, there is no β satisfying both 1st and 2nd conditions above.

Then, it is quite difficult to calculate the observable $\langle O \rangle_f$ by using the modification formula (7). This problem occurs also in the ordinary reweighting technique [26]. Although the modification method is different method from the reweighting method, the modification method has the same problem as the reweighting method.

Therefore, the alternative way is desired to calculate the observable $\langle O \rangle_f$ without direct calculation of the reweighting factor $\langle f/g \rangle_g$, in particular, the quantity $\langle f \rangle$. In the next subsection, we develop such method, namely the multi-modification method, and demonstrate that

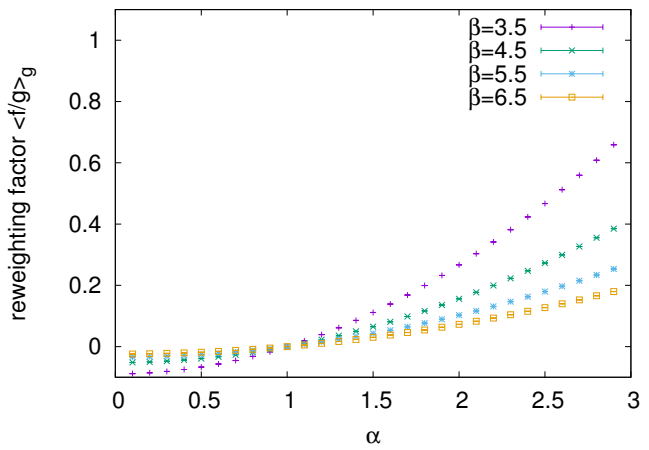

FIG. 5. The reweighting factor $\langle f/g\rangle_g$ calculated in the g-theory by the complex Langevin method with $\beta = 3.5, 4.5, 5.5, 6.5$ plotted against α.

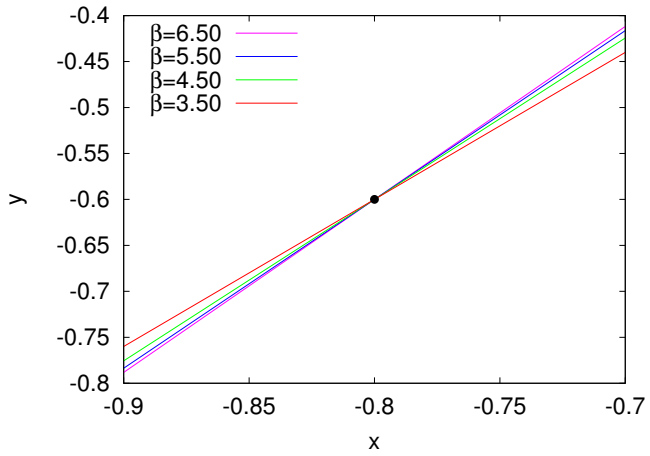

FIG. 6. A set of lines defined in Eq. (10) with $\alpha = 1.5$ and $\beta = 3.5, 4.5, 5.5, 6.5$. The analytic solution is substituted for the coefficients a_g and b_g. The black circle denotes the analytic solution of (x, y).

the method works well in the Gaussian model.

B. multi-modification method

We demonstrate that the multi-modification method discussed in Sec. II gives the correct results in the Gaussian model even in the parameter-region where the sign problem is severe.

As discussed in Sec. II, we rewrite the modification formula (7) to the linear function as Eq. (10). We again use the modification function $g(x)$ defined in Eq. (21), and each linear function (10) is determined once $g(x)$ is fixed. Our approach is to calculate the quantities $\langle f\rangle$ and $\langle O\rangle_f$, namely x and y, as the intersection point of a set of lines $\{y = a_g x + b_g\}$ by calculating a_g and b_g for several β.

Ideally, a set of lines $\{y = a_g x + b_g\}$ has an unique intersection point shown as Fig. 6. However, the coefficients a_g and b_g have errors because the actual calculation for the observables $\langle O\rangle_{f+g}$, $\langle O\rangle_g$ and $\langle g\rangle$ are performed by the complex Langevin method. Thus the value of y has an error for each x due to the errors of the coefficients and the possible values (x, y) form a band-like region in xy-plane. We estimate the correct values of x and y, namely $\langle f\rangle$ and $\langle O\rangle_f$, from the overlap of all the band-like regions corresponding several β.

To apply the multi-modification method, we have to choose appropriate function $g(x)$, namely the parameter β here, so that the following quantities can be correctly calculated:

1. The observables $\langle O\rangle_g$ and $\langle O\rangle_{f+g}$
2. The average $\langle g\rangle$

Note that the 2nd condition in the multi-modification method is weaker than the 2nd condition in the original modification method while the 1st conditions are same. In principle, the average $\langle g(z)\rangle$ in the quenched theory is always calculable by the Monte Carlo method because $S_{\rm q}(x)$ is supposed to be real. However, this quantity becomes difficult to calculate if the modification function $g(x)$ has violent oscillation on its phase. Thus $g(x)$ should be chosen so that the sign of $\mathrm{Re}(g(x))$ is not frequently changed in the importance sampling of the Monte Carlo simulation.

Actual condition for β is investigated as follows. From the 1st condition, β is constrained to $3.5 \leq \beta$ as we already know in the previous subsection. Fortunately, when $3.5 \leq \beta$, the quantity $\langle g\rangle$ can be correctly obtained by the Monte Carlo method. In fact, the sign of $\mathrm{Re}(g(x))$ is almost always negative in the Monte Carlo calculation if $3.5 \leq \beta$. This result reflects the fact that the sign problem in the Gaussian model is not severe when $3.5 \leq \beta$. Therefore, if $3.5 \leq \beta$, the above conditions are satisfied. In other words, the coefficients a_g and b_g in Eq. (10) can be correctly calculated when $3.5 \leq \beta$.

In our analysis, 6 values of β are taken from 3.5 to 8.5. With those β, the observables $\langle O\rangle_{f+g}$ and $\langle O\rangle_g$ are calculated by the complex Langevin method, and the quantity $\langle g\rangle$ is calculated by the Monte Carlo method. In the complex Langevin method, we also use the Euler's method to solve the complex Langevin equation for the total Langevin flow time 10^7 with the discrete time $dt = 10^{-2}$. We take configuration every 1 Langevin time after 10^2 Langevin time for the thermalization. We consider $O(x) = x^2$ as the observable, and then $\mathrm{Re}(z^2)$ is calculated by the multi-modification method.

In Fig. 7, we show the regions of the possible values of (x, y) for each β when $O(z) = \mathrm{Re}z^2$ with $\alpha = 1.5$. The black square at $(-0.8, -0.6)$ denotes the analytic solution of $(1/\langle f\rangle, \langle \mathrm{Re}z^2\rangle_f)$. The gray region is the overlap

of all the regions for each β, and certainly covers the analytic solution. We have performed the similar analysis for the other α on $O(z) = \mathrm{Re}z^2$. In Fig. 8, we show the numerical results of $O(z) = \mathrm{Re}z^2$ for each α calculated by the multi-modification method. In addition to the parameter-region where the sign problem is not severe, the multi-modification method certainly reproduces the correct results even in the parameter-region where the sign problem is severe and the original complex Langevin method fails to give the correct results.

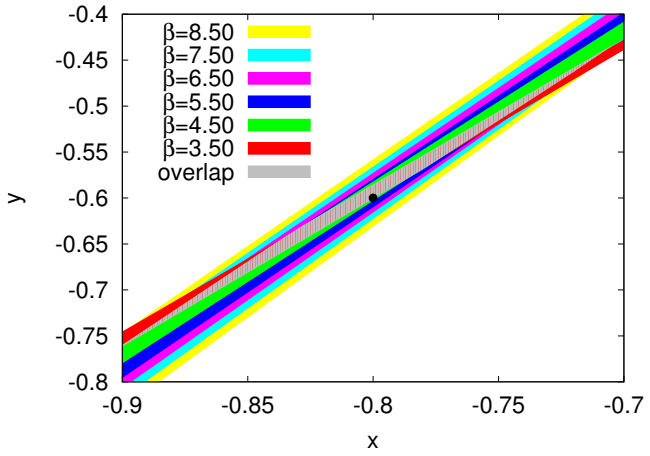

FIG. 7. The region of the possible values of (x, y) from Eq. (10) for each β and their overlap (gray colored) when $O(z) = \mathrm{Re}z^2$ and $\alpha = 1.5$. The black circle at $(-0.8, -0.6)$ is the point of the analytic solution.

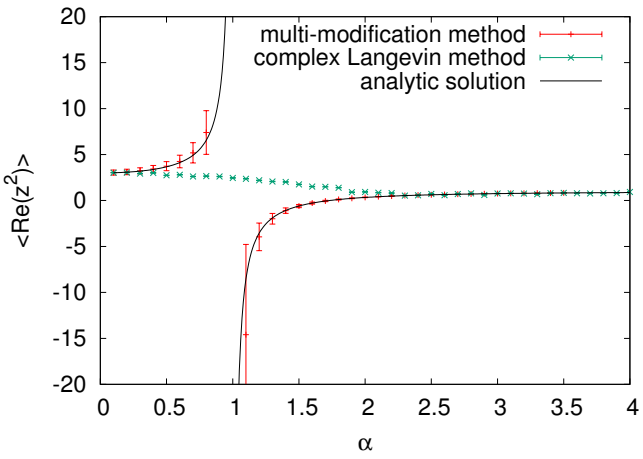

FIG. 8. The numerical results of $O(z) = \mathrm{Re}z^2$ with $0 < \alpha < 3$ by the multi-modification method.

C. Discussion

The multi-modification method has some advantages than the original modification method. In our method, we do not have to directly calculate $\langle f \rangle$, which is the average of the oscillatory function. It is difficult to calculate the quantity with enough precision, in particular, when the sign problem is severe. However, in our method, both $\langle f \rangle$ and $\langle O \rangle_f$ can be simultaneously obtained without direct calculation of $\langle f \rangle$ itself. In fact, the numerical results of the average $\langle f \rangle$ obtained in our method well reproduces the correct results, as shown in Fig. 9.

Moreover, as another advantage of the multi-modification, we can reduce the errors of the observable $\langle O \rangle_f$ by considering more modification functions $g(x)$. In particular, if the coefficients a_g and b_g are quite different between considered functions $g(x)$, the overlap region is more restricted. Then, in principle, we can improve the numerical calculation by considering many modifications of the theory. To that end, we have to develop more systematic ways to find an appropriate function $g(x)$ as well as the function defined in Eq. (21). However, it is work in progress.

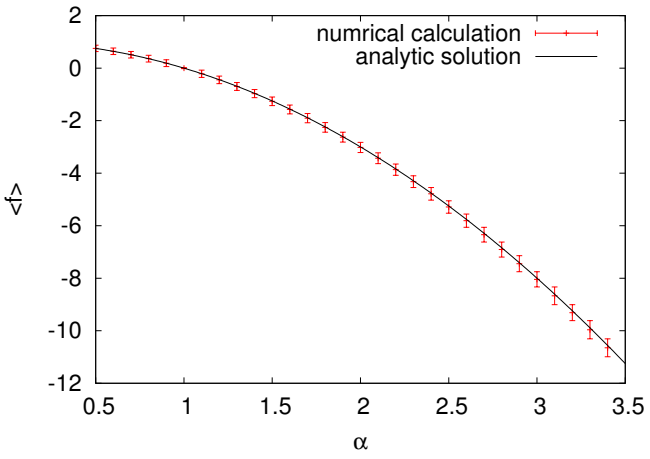

FIG. 9. The numerical results of $\langle f \rangle$ for each α by the multi-modification method.

In addition to $O(z) = \mathrm{Re}z^2$, we have performed similar analysis on other observables with higher power of z, $O(z) = \mathrm{Re}z^4$ and $O(z) = \mathrm{Re}z^6$. In Figs. 10 and 11, the numerical results for each observable are shown. Although the results cover the analytical solutions with their error, its errors are larger by increasing the power of z.

From Figs. 8, 10 and 11 One can see that the numerical error becomes larger around $\alpha = 1$. This is because the changes of the coefficients a_g and b_g are small even if β changes when $\alpha \sim 1$. As a result, the overlap of the linear functions forms a wide region, then the numerical error for the observable $\langle O \rangle_f$ becomes larger.

In our method, the observable in the original theory

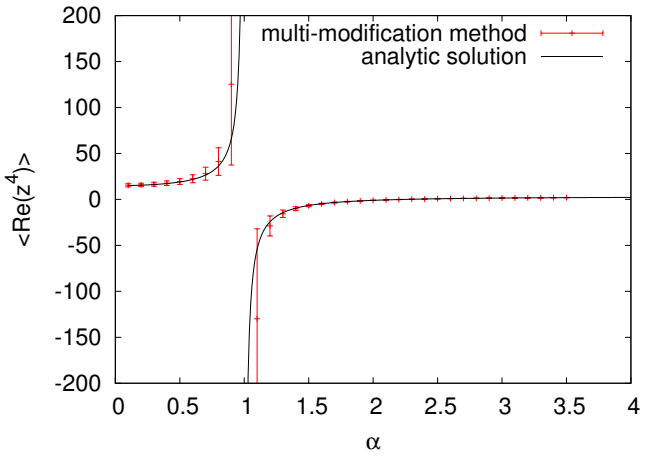

FIG. 10. The numerical results of $O(z) = \mathrm{Re}z^4$ with $0 < \alpha < 3$ by the multi-modification method.

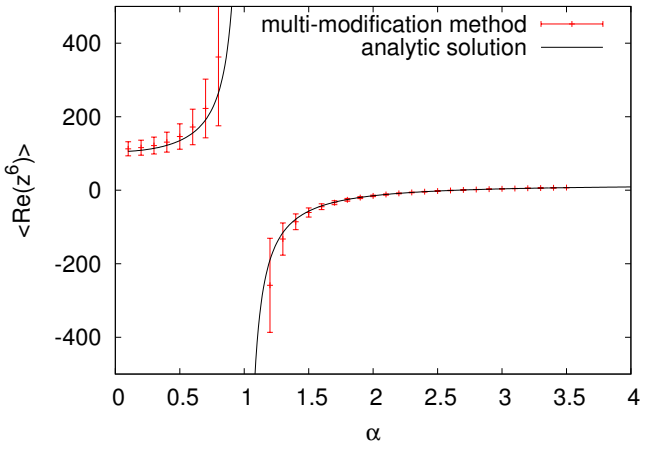

FIG. 11. The numerical results of $O(z) = \mathrm{Re}z^6$ with $0 < \alpha < 3$ by the multi-modification method.

where the sign problem is severe is obtained from the observables in the modified and reference theories where the sign problem is not severe. This concept is similar to that in the reweighting method. In fact, when we directly apply the modification method and the sign problem is severe, the reweighting factor $\langle f/g \rangle_g$ is small and the modification has the same difficulty as the reweighting method. However, the modification method is a different method from the reweighting method. In particular, one does not have to calculate the reweighting factor in our method proposed in this paper. Moreover, in our method, there is no overlap problem which appears in the reweighting method[31].

In this study, we adopt the complex Langevin method as a calculating tool of each observable in the theories with the complex action. As other tools instead of the complex Langevin method, there are some candidates proposed in Ref. [32, 33, 34, 35, 36, 37, 38, 39, 40]. Basically, they are based on the Lefschetz thimble method [41, 42] If one of them is adopted, the appropriate range of β is different from the case of the complex Langevin method, $3.5 \leq \beta$. It is nontrivial which method is superior as the actual calculating tool. The reason why we adopt the complex Langevin method is that the correctness of the complex Langevin method is easy to judge by investigating the distribution of the drift term, at least in this model.

Finally, we mention the applicable scope of the multi-modification method. To calculate the observable $\langle O(z) \rangle_f$ in the original theory by the multi-modification method, the 3 quantities, $\langle g(x) \rangle$, $\langle O(x) \rangle_g$ and $\langle O(x) \rangle_{f+g}$ should be calculated. If one can find some appropriate modification functions $g(x)$ satisfying the conditions, the observable $\langle O(z) \rangle_f$ is always obtained by the procedure demonstrated above. Therefore, our method can be applied the higher-dimensional theories. For example, some interested theories such as the thirring model, chiral random matrix theories and QCD have the sign problem at finite chemical potential. In those theories, there are some parameter region where the complex Langevin method gives the correct results. From the results in the parameter region, one can apply the multi-modification method and calculate the observable in the region where the sign problem is severe and the complex Langevin method gives the incorrect results.

IV. SUMMARY

In this paper, we have developed a way named multi-modification method to solve the sign problem by improving our previous method [25]. In our method, instead calculating an observable in the original theory with the fermion determinant $f(x)$, one calculates the observables $\langle O(x) \rangle_g$ and $\langle O(x) \rangle_{f+g}$ in the theories with the fermion determinants $g(x)$ and $f(x)+g(x)$ and the average $\langle g(x) \rangle$ in the quenched theory. If one can find the appropriate functions $g(x)$, the observable in the original theory can be reconstructed from the overlap of the linear functions defined in Eq. (10) for each $g(x)$.

By applying it to a toy model, the Gaussian model, we demonstrate how our method works when the complex Langevin method is adopted as a computational scheme. Although the modification function $g(x)$ is arbitrary in general, we consider only the function which has the same form as $f(x)$ and different parameter β defined in Eq. (21). As a result, the correct results are correctly reproduced by our method in the parameter region where the sign problem is severe and the direct use of the complex Langevin method results in the wrong convergence. If other appropriate function $g(x)$ exists and one can find it, the error of the observable will be smaller. To develop more systematic ways to find an appropriate function $g(x)$ is work in progress.

Since it is not difficult, at least formally, to generalize

our method to higher dimensional problem, we would like to apply our method to the higher dimensional theories, such as the Thirring model, the random matrix theory, and finally QCD.

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