Laplace operators on Sasaki-Einstein manifolds

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We decompose the de Rham Laplacian on Sasaki-Einstein manifolds as a sum over positive definite terms. An immediate consequence is a lower bound on its eigenvalue spectrum. The resulting inequality constitutes a supergravity equivalent of the unitarity bounds in dual superconformal field theories. The proof relies on a generalization of Kähler identities to the Sasaki-Einstein case.

I. INTRODUCTION

A textbook result in Kähler geometry relates the de Rham with the Dolbeault Laplacian, $\Delta = 2\Delta_{\bar\partial}$. The main result of this note is the derivation of a similar identity in the case of Sasaki-Einstein manifolds:

$$\begin{aligned}\Delta &= 2\Delta_{\bar\partial_B} - \pounds_\xi^2 - 2\imath(n-d^0)\pounds_\xi \\ &\quad + 2L\Lambda + 2(n-d^0)L_\eta\Lambda_\eta + 2\imath(L_\eta\bar\partial_B^* - \bar\partial_B\Lambda_\eta).\end{aligned} \tag{1}$$

The right hand side features the tangential Cauchy-Riemann operator, the Lefschetz operator, the action of the Reeb vector, as well as their adjoints. Full definitions will be given shortly. $\Delta = 2\Delta_{\bar\partial}$ can be derived from the Kähler identities, commutators between the Dolbeault and Lefschetz operators and their adjoints [1, 2]. Our proof will follow a similar route by obtaining Kähler-like identities that hold on Sasaki-Einstein manifolds. These are summarized in appendix A.

Both equation (1) as well as the identities in appendix A find application in the AdS/CFT correspondence. Freund-Rubin compactification on Sasaki-Einstein manifolds yields supergravity duals of superconformal field theories ([3] and references therein). The AdS/CFT dictionary links the conformal energy of SCFT operators to the spectrum of Δ, their R-charge to that of the Lie-derivative along the Reeb vector, $\pounds_\xi$. The conformal energy, R-charge, and spin of any SCFT operator have to satisfy the unitarity bounds [4, 5], which should be reflected on the supergravity side in the spectrum of Δ. We will argue in section III that equation (1) allows us to re-derive the unitarity bounds from supergravity when considered in conjunction with the calculations in [6, 7].

Furthermore, the Kähler-like identities allow for a study of the eigenmodes of Δ. In the case where the Sasaki-Einstein manifold has a coset structure, this has been done using harmonic analysis [8]. [6, 7] obtained the structure of the Kaluza-Klein spectrum of generic Sasaki-Einstein manifolds using a construction similar to that in [9], which can be nicely summarized in terms of the identities in appendix A: Given any eigen k-form ω of Δ, one diagonalizes the action of Δ on the $k+1$-forms $\{\partial_B\omega, \bar\partial_B\omega, L_\eta\omega, L\omega, \partial_B\bar\partial_B\omega, \dots\}$. The resulting eigenstates fill out representations of the superconformal algebra, the Kohn-Rossi cohomology groups correspond to short multiplets. Whereas the original calculations were based on a rather tiresome direct approach, the methods developed in this note are expected to simplify that kind of anlysis considerably.

A further application of (1) is the stability analysis of Pilch-Warner solutions by Pilch and Yoo [10]. In the absence of general theorems concerning Laplace operators on Sasaki-Einstien manifolds, the authors constructed explicitly $(1,1)$-forms whose existence renders these solutions perturbatively unstable.

The tangential Cauchy-Riemann operator $\bar\partial_B$ and the associated Kohn-Rossi cohomology groups $H^{p,q}_{\bar\partial_B}(S)$ were first introduced in [11, 12]. Given a complex manifold with boundary, Lewy, Kohn, and Rossi considered under what circumstances functions on the boundary can be extended to holomorphic functions in the bulk. Clearly they have to satisfy the projection of the Cauchy-Riemann equations onto the boundary, hence the name for $\bar\partial_B$. The Kohn-Rossi cohomology groups feature also in the work by Yau and collaborators on the complex plateau problem [13–15]. This problem concerns the question when a real manifold is also the boundary of a complex manifold.

Section II gives a full proof of (1) after setting the stage by giving all necessary definitions. Since the proof is based on the equivalent considerations in the Kähler case, our discussion will follow [1, 2] very closely. We will comment on further applications of both equation (1) and the identities in section III.

II. KÄHLER-LIKE IDENTITIES

A. Exterior calculus on Sasaki-Einstein manifolds

Consider a $d = 2n+1$ dimensional Sasaki-Einstein manifold S. Given the Reeb vector ξ and the contact form η, the tangent bundle splits as $TS = D \oplus L_\xi$.[1] Furthermore, there is a two-form $J = \frac{1}{2}d\eta$ with $i_\xi J = 0$.

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[1] L_ξ is the line tangent to ξ. In what follows we will set $L_\xi = \xi$ and $L_\xi^* = \eta$. See section 1 of [16] for a review of Sasaki-Einstein geometry.

J defines an endomorphism on TS which satisfies $J^2 = -1 + \xi \otimes \eta$. Since $\eta(D) = 0$, one can decompose the complexified tangent bundle as $T_{\mathbb{C}}S = (\mathbb{C} \otimes D)^{1,0} \oplus (\mathbb{C} \otimes D)^{0,1} \oplus (\mathbb{C} \otimes \xi)$. This in turn induces a corresponding decomposition on the complexified cotangent bundle

$$T^*_{\mathbb{C}}S = \Omega^{1,0} \oplus \Omega^{0,1} \oplus (\mathbb{C} \otimes \eta), \tag{2}$$

which also extends to the exterior algebra

$$\Omega^*_{\mathbb{C}} = \bigoplus_{p,q} \Omega^{p,q} \wedge (1 \oplus \eta). \tag{3}$$

Elements of $\Omega^*_{\mathbb{C}}$ that vanish under the action of i_ξ are called *horizontal*, while those annihilated by $\eta\wedge$ are *vertical*.

The decomposition (2) induces a decomposition of the exterior derivative,

$$d = \partial_B + \bar{\partial}_B + \pounds_\xi \eta \wedge . \tag{4}$$

∂_B and $\bar{\partial}_B$ are the *tangential Cauchy-Riemann operators*. They satisfy $\{\partial_B, \bar{\partial}_B\} = -2J \wedge \pounds_\xi$ as well as $\partial_B^2 = \bar{\partial}_B^2 = 0$. The sequence

$$\dots \xrightarrow{\bar{\partial}_B} \Omega^{p,q-1} \xrightarrow{\bar{\partial}_B} \Omega^{p,q} \xrightarrow{\bar{\partial}_B} \Omega^{p,q+1} \xrightarrow{\bar{\partial}_B} \dots$$

gives rise to the *Kohn-Rossi cohomology groups* $H^{p,q}_{\bar{\partial}_B}(S)$. Continuing with the theme of generalizing concepts from Kähler geometry to Sasaki-Einstein manifolds, we define the *Lefschetz operator* $L : \Omega^k_{\mathbb{C}} \to \Omega^{k+2}_{\mathbb{C}}$ via $\alpha \mapsto J \wedge \alpha$ and the *Reeb operator* $L_\eta : \Omega^k_{\mathbb{C}} \to \Omega^{k+1}_C$ as $\alpha \mapsto \eta \wedge \alpha$.

Introducing the *Hodge star*[2]

$$\star\bar{\alpha} \wedge \beta = \frac{1}{p!} \bar{\alpha}^{m_1 \dots m_p} \beta_{m_1 \dots m_p} \,\mathrm{vol} = \langle \alpha, \beta \rangle \,\mathrm{vol},$$

allows us to define adjoints for the above operators when acting on $\Omega^k_{\mathbb{C}}$:

$$\begin{aligned} d^* &= (-1)^k \star d \star, \\ \partial_B^* &= (-1)^k \star \bar{\partial}_B \star, \\ \bar{\partial}_B^* &= (-1)^k \star \partial_B \star, \\ \Lambda &= L^* = \star L \star = J \lrcorner, \\ \Lambda_\eta &= L_\eta^* = (-1)^{k+1} \star L_\eta \star = i_\xi, \\ (L_\eta \pounds_\xi)^* &= -\Lambda_\eta \pounds_\xi. \end{aligned} \tag{5}$$

Recall that on odd-dimensional manifolds $\star$ satisfies $\star\star = 1$.

When restricted to D, the action of J becomes that of an *almost complex structure* $\mathcal{I}$ which acts as $\mathcal{I}(\alpha) = J_m{}^n \alpha_n dx^m$ and $\mathcal{I}(X) = X^m J_m{}^n \partial_n$. Of course $\Omega^{1,0} = \{\alpha \in \Omega^1 | \mathcal{I}(\omega) = \imath\omega\}$. We also define

$$\boldsymbol{I} = \sum_{p,q} \imath^{p-q} \Pi^{p,q},$$

which makes use of the projection $\Pi^{p,q} : \Omega^*_{\mathbb{C}} \to \Omega^{p,q}$.

It will turn out useful to distinguish between the rank of a form on $\Omega^*_{\mathbb{C}}$ and on $\bigwedge^* D^*$. Hence we define the operator d^0 on $\Omega^*_{\mathbb{C}}$ via

$$d^0|_{\bigwedge^k D^* \wedge (1 \oplus \eta)} = k \cdot \mathrm{id}\,.$$

By definition, d^0 commutes with L_η. A first example of the uses of d^0 is given by the notion of *primitive forms.* $\alpha \in \Omega^*_{\mathbb{C}}$ with $d^0 \leq n$ is primitive if and only if $\Lambda\alpha = 0$. Essentially the idea of primitivity on $\bigwedge^* D^*$ is the same as on Kähler manifolds, the contact one-form just comes along for the ride and there is in principle no difference between horizontal and vertical forms. We define P^k as the set of primitive elements of $\bigwedge^k D^*$.

Next we introduce an orthonormal frame e^i on D^*. Defining $z^i = e^{2i-1} + \imath e^{2i}$ and imposing $\mathcal{I}(z^i) = \imath z^i$, consistency requires that $\mathcal{I}(e^{2i-1}) = -e^{2i}$ and $\mathcal{I}(e^{2i}) = e^{2i-1}$. Then

$$J = \sum_i^n e^{2i-1} \wedge e^{2i} = \frac{\imath}{2} \sum_i^n z^i \wedge \bar{z}^i.$$

Defining $e^{2n+1} = \eta$, one finds $\mathrm{vol} = \mathrm{vol}_{D^*} \wedge e^{2n+1} = \frac{1}{n!} J^n \wedge \eta$.

In what follows, we will make use of two results concerning the Hodge star. To begin, assume that $(V, \langle,\rangle)$ is a Euclidean vector space admiting a decomposition $V = W_1 \oplus W_2$ that is compatible with the metric $\langle,\rangle$. For simplicity we assume that $\dim_{\mathbb{R}} W_i \in 2\mathbb{Z}$. The metrics $\langle,\rangle_i$ induce Hodge star operators $\bullet_i$, $i = 1, 2$. Then $\bigwedge^* V^* = \bigwedge W_1^* \otimes \bigwedge W_2^*$, and for $\alpha_i \in \bigwedge^{k_i} W_i^*$, the Hodge dual on $\bigwedge^* V^*$, $\bullet$, threads as

$$\bullet\,(\alpha_1 \otimes \alpha_2) = (-1)^{k_1 k_2} \bullet_1 \alpha_1 \otimes \bullet_2 \alpha_2, \tag{6}$$

since ($\beta_i \in W_i$)

$$\begin{aligned} &\bullet\,(\alpha_1 \otimes \alpha_2) \wedge (\beta_1 \otimes \beta_2) = \langle \alpha_1, \beta_1 \rangle_1 \langle \alpha_2, \beta_2 \rangle_2 \,\mathrm{vol}_1 \,\mathrm{vol}_2 \\ &= (-1)^{k_1 k_2} \bullet_1 \alpha_1 \wedge \bullet_2 \alpha_2 \wedge \beta_1 \wedge \beta_2. \end{aligned}$$

One can use identical considerations to decompose the action of $\star$ on $\Omega^*_{\mathbb{C}}$ into seperate operations on D^* and η. Introducing a hodge dual $\bullet$ on D^*, one finds

$$\star\,|_{\bigwedge^* D^*} = L_\eta \bullet, \qquad \star|_{\bigwedge^* D^* \wedge \eta} = \bullet(-1)^{d^0} \Lambda_\eta. \tag{7}$$

B. Lefschetz decomposition

The starting point for our discussion of Lefschetz decomposition is the commutator

$$[L, \Lambda] = (d^0 - n). \tag{8}$$

[2] In components

$$\star\alpha_{m_1 \dots m_p} = \frac{\sqrt{g}}{p!} \epsilon_{m_1 \dots m_{d-p}}{}^{n_1 \dots n_p} \alpha_{n_1 \dots n_p}.$$

The proof is via induction in n. Consider $d = 3$, $n = 1$. Then $\Omega^*_{\mathbb{C}}$ is spanned by $\{1, \eta, \mu_i, J, J \wedge \eta\}$ where $\mu_i \in \Omega^{1,0} \oplus \Omega^{0,1}$ and both μ_i are annihilated by L and Λ. Then $\Lambda J = 1$ and thus $[L,\Lambda]|_{\Omega^0_{\mathbb{C}}} = -1$, $[L,\Lambda]|_\eta = -1$, $[L,\Lambda]|_{D^*=0}$, $[L,\Lambda]|_{\Omega^{1,1}} = 1$, and $[L,\Lambda]|_{\Omega^3_{\mathbb{C}}} = 1$. Hence

$$[L,\Lambda]|_{\bigwedge^k D^* \wedge (1\oplus\eta)} = (k-1), \qquad k = 0,1,2,$$

as claimed. The induction then proceeds as in [1]

(8) generalizes to

$$[L^i,\Lambda]|_{\bigwedge^k D^* \wedge (1\oplus\eta)} = \imath(k-n+i-1)L^{i-1}. \tag{9}$$

Again the proof is a copy of that in [1].

To proceed we follow [2]. Restricting to $\bigwedge^* D^*$ one can copy all results from proposition 6.20 to lemma 6.24. The most important of these results is *Lefschetz decomposition.* Given $\alpha \in \bigwedge^k D^*$, there is a unique decomposition

$$\alpha = \sum_r L^r \alpha_r, \qquad \alpha \in P^{k-2r}.$$

The decomposition is compatible with the bidigree decomposition and with the decomposition into horizontal and vertical components. Moreover,

$$L^{n-k} : \bigwedge^k D^* \to \bigwedge^{2n-k} D^* \tag{10}$$

is an isomorphism and the primitivity condition is equivalent to $L^{n-k+1}\alpha = 0$.

The Lefschetz decomposition becomes incredibly useful when used together with the Bidigree decomposition, equation (7) and the identity

$$\begin{aligned} \forall \alpha \in P^k, \quad & \bullet L^j \alpha = F(n,j,k) L^{n-k-j} \boldsymbol{I}(\alpha), \\ & F(n,j,k) = (-1)^{\frac{k(k-1)}{2}} \frac{j!}{(n-k-j)!}. \end{aligned} \tag{11}$$

Since no differential operators are involved and $\alpha \in \bigwedge^k D^*$, one can copy the proof in [1] after adjusting for conventions. Once the dust settles, the only difference is in the k-dependent prefactor.

C. Calculating the identities

We are finally in a position to make use of the previous results and calculate the (anti-) commutators. The results are in summarized in table I. A number of identities are fairly obvious:

$$\begin{aligned} 0 &= [\partial_B, L] = [\bar\partial_B, L] = [\partial_B^*, \Lambda] = [\bar\partial_B^*, \Lambda] \\ &= [L, L_\eta] = [\Lambda, \Lambda_\eta] = [L_\eta, \Lambda]. \end{aligned}$$

One finds $\{L_\eta, \Lambda_\eta\} = 1$ by direct calculation using the decomposition $\alpha = \alpha_H + L_\eta \alpha_V$. Finally, $[d^0, \partial_B] = \partial_B + L\Lambda_\eta$.

The most involved calculation is that of the commutator

$$[\Lambda, \bar\partial_B] = -\partial_B^* + \imath L_\eta \Lambda + (n - d^0)\Lambda_\eta. \tag{12}$$

Before we turn to the proof, let us try to interpret this result as a generalization of the Kähler case $[\Lambda, \bar\partial] = -\imath\partial^*$. The naive guess $[\Lambda, \bar\partial_B] \stackrel{?}{=} -\imath\partial_B^*$ cannot be correct since the left hand side maps $[\Lambda, \bar\partial_B] : \bigwedge^* D^* \to \bigwedge^* D^*$ while $\partial_B^* : \bigwedge^* D^* \to \bigwedge^* D^* \wedge (1 \oplus \eta)$. Similarly, the right hand side annihilates η while the left hand side does not. One can guess the correct result by considering the action of both sides on J and η, adding suitable terms on the right hand side to achieve equality.

The proof of (12) is once again an elaboration on the proof for Kähler manifolds in [1]. Let us first consider horizontal forms. Here, it is sufficient to explicitly evaluate the action of (12) $L^i\alpha$ for $\alpha \in P^k$; the result will generalize for generic elements of $\bigwedge^* D^*$ due to Lefschetz decomposition. Furthermore one applies Lefschetz decomposition to $\bar\partial_B \alpha = \alpha_0 + L\alpha_1 + L^2\alpha_2 + \ldots$. We have $\alpha \in P^k$ and thus $0 = \sum_j L^{n-k+1+j}\alpha_j$ and finally $L^{n-k+1+j}\alpha_j = 0$. Using equation (10) it follows that most of the α_j vanish and $\bar\partial_B\alpha = \alpha_0 + L\alpha_1$.

Using (9) one finds

$$[\Lambda, \bar\partial_B] L^i \alpha = -iL^{i-1}\alpha_0 - (k+i-n-1)L^i\alpha_1.$$

Similarly, using $\bar\partial_B \boldsymbol{I}(\alpha) = \imath \boldsymbol{I}(\bar\partial_B\alpha)$ and $\boldsymbol{I}^2(\bigwedge^k D^*) = (-1)^k$ as well as (7) and (11)

$$\star\bar\partial_B \star L^i\alpha = \imath(-1)^{k^2}[\Lambda, \bar\partial_B]L^i\alpha - (-1)^k L_\eta [L^i, \Lambda]\alpha.$$

Finally,

$$[\Lambda, \bar\partial_B]|_{\bigwedge^* D^*} = -\imath\partial_B^* + \imath L_\eta \Lambda.$$

To study vertical forms, we consider $L_\eta L^i \alpha$. Again $\alpha \in P^k$ and $\bar\partial_B\alpha = \alpha_0 + L\alpha_1$. Then

$$\begin{aligned} [\Lambda, \bar\partial_B] L_\eta L^i \alpha = iL_\eta L^{i-1}\alpha_0 &+ (k+i-n-1)L_\eta L^i \alpha_1 \\ &+ [n - (2i+k)]L^i\alpha. \end{aligned}$$

Note that $2i+k$ is the degree of $L^i\alpha$. Furthermore,

$$\begin{aligned} \star\bar\partial_B \star L_\eta L^i \alpha = (-1)^{k^2+1}\imath & \\ \times [iL_\eta L^{i-1}\alpha_0 &+ (k+i-n-1)L_\eta L^i\alpha_1]. \end{aligned}$$

In total,

$$[\Lambda, \bar\partial_B](L_\eta L^i \alpha) = \{-\imath\partial_B^* + [n - (2i+k)]\Lambda_\eta\}(L_\eta L^i\alpha).$$

Since $L_\eta\Lambda(L_\eta L^i\alpha) = 0$, we can add or subtract $\imath L_\eta\Lambda$. Therefore it is consistent to combine the results on horizontal and vertical forms into the overall result (12). An identical calculation or complex conjugation give $[\Lambda, \partial_B]$. This completes the proof.

We can compute the computator of the adjoints ($\alpha \in \Omega^p_{\mathbb{C}}$):

$$[L, \partial_B^*]\alpha = (-1)^p[-\imath \star \partial_B^* \star + \imath \star L_\eta \Lambda \star + \star (n - d^0)\Lambda_\eta \star]\alpha_p.$$

With $\star(n-d^0)\star = (d^0-n)$, $\star\partial_B^*\star\alpha = (-1)^{p+1}\bar\partial_B\alpha$, and $\star L_\eta\Lambda\star\alpha = (-1)^{p+1}\Lambda_\eta L\alpha$ one finds

$$[L,\partial_B^*] = \imath\bar\partial_B - \imath\Lambda_\eta L + (d^0-n)L_\eta,$$
$$[L,\bar\partial_B^*] = -\imath\partial_B + \imath\Lambda_\eta L + (d^0-n)L_\eta.$$

The calculation of the anticommutator $\{\Lambda_\eta,\bar\partial_B\}$ is considerably simpler. Consider again $\alpha\in P^k$ with $\bar\partial_B\alpha = \alpha_0 + L\alpha_1$. Then $\Lambda_\eta\bar\partial_B\alpha = 0$ and $\bar\partial_B\Lambda_\eta\alpha = 0$. The next step is only slightly more complicated: $\Lambda_\eta\bar\partial_B L_\eta\alpha = -\bar\partial_B\alpha$, $\bar\partial_B\Lambda_\eta L_\eta\alpha = \bar\partial_B\alpha$ and thus $\{\Lambda_\eta,\bar\partial_B\} = 0$. Similarly $\{\Lambda_\eta,\partial_B\} = 0$ as well as the extension to the adjoint case.

This concludes the calculation of the identities. The (anti-) commutators allow us to express $\Delta = d^*d + dd^*$ in terms of $\Delta_{\bar\partial_B} = \bar\partial_B^*\bar\partial_B + \bar\partial_B\bar\partial_B^*$. The decomposition (4) yields

$$\Delta = \Delta_{\partial_B} + \Delta_{\bar\partial_B} + \{\partial_B,\bar\partial_B^*\} + \{\bar\partial_B,\partial_B^*\} - \pounds_\xi^2.$$

Then, using $\{\partial_B,\bar\partial_B^*\} = \{\partial_B, L_\eta\Lambda\} + \imath\partial_B\Lambda_\eta$ one shows that

$$\Delta_{\partial_B} = \Delta_{\bar\partial_B} - 2\imath(n-d^0)\pounds_\xi + \{\partial_B - \bar\partial_B, L_\eta\Lambda\} - \imath(\partial_B+\bar\partial_B)\Lambda_\eta,$$

which leads to

$$\Delta = 2\Delta_{\bar\partial_B} - 2\imath(n-d^0)\pounds_\xi - \pounds_\xi^2 + 2\{\partial_B, L_\eta\Lambda\} - 2\imath\bar\partial_B\Lambda_\eta.$$

Application of $\{\partial_B, L_\eta\Lambda\} = \imath L_\eta\bar\partial_B^* + (n-d^0)L_\eta\Lambda_\eta + L\Lambda$ completes the proof of (1).

III. DISCUSSION

We turn to the spectral problem for Δ. Consider a k-form ω with $\pounds_\xi\omega = \imath q$, $q\geq 0$, and $d^0\leq n$. Clearly all terms on the right hand side of (1) are positive definite except for the mixed term $M = \imath(L_\eta\bar\partial_B^* - \bar\partial_B\Lambda_\eta) = N + N^*$. M is self-adjoint and its spectrum is real. Moreover, $N^2 = 0$ and $N(\bigwedge^* D^*)\subset\bigwedge^* D^*\wedge\eta$ and $N(\bigwedge^* D^*\wedge\eta) = 0$. That is, N maps horizontal to vertical forms and annihilates the latter. N^* behaves accordingly and it follows that $\langle\omega, M\omega\rangle$ vanishes if ω is horizontal or vertical. This is also the case if ω is neither horizontal nor vertical yet holomorphic.[3] As long as we restrict to one of these cases, (1) takes the form of a bound on the spectrum of Δ.

This was conjectured and partially shown in the context of the calculations of the superconformal index in [6, 7]. Here, the spectrum was constructed from primitive elements of $\Omega^{p,q}$. For such forms, (1) clearly implies

$$\Delta \geq q^2 + 2q(n-d^0) \tag{13}$$

with equality if and only if $\bar\partial_B\omega = \bar\partial_B^*\omega = 0$. In the Kähler case, the latter of these is implied by transversality — $d^*\omega = 0$. Here however, $d^*\omega = 0$ leads only to the vanishing of the horizontal component of $\bar\partial_B^*\omega$. Indeed,

$$\partial_B^*\omega = \imath L_\eta\Lambda\omega,\quad \bar\partial_B^*\omega = -\imath L_\eta\Lambda\omega,$$

which vanishes since ω was assumed to be primitive. Assuming that every element of $H^{p,q}_{\bar\partial_B}(S)$ has a representative closed under $\bar\partial_B^*$, the bound (13) is saturated on the elements of $H^{p,q}_{\bar\partial_B}(S)$. These are the forms that correspond to the short multiplets in the SCFT, and (13) together with the expressions for the derived eigenmodes of Δ given in [6, 7] allows to recover the unitarity bounds from supergravity. Note that (13) and a precursor to (1) were already conjectured in those references. Note that the appendix of [7] contains an argument that every element of $H^{p,q}_{\bar\partial_B}(S)$ is either primitive, carrying zero charge, or both. For the cases of interest in the context of that paper it turned out that all elements are primitive.

Since we found Sasaki-Einstein equivalents of both $\Delta = 2\Delta_{\bar\partial}$ and the Kähler identities, it is tempting to ask how much more of Kähler geometry can be generalized. For example, since $\Delta_{\bar\partial}$ is self-adjoint and elliptic, one can show that $\Omega^k_{\mathbb{C}} = \mathcal{H}^k\oplus\Delta_{\bar\partial}(\Omega^k_{\mathbb{C}})$ which implies Hodge's theorem. Similarly, the relation between the de Rham and Hodge Laplacians allows for an isomorphism between the respective spaces of harmonic forms. However, it turns out that $\Delta_{\bar\partial_B}$ is not elliptic. We will sketch the calculation leading to this result. Recall that $\Delta_{\bar\partial_B}$ is elliptic if the symbol $\sigma_{\Delta_{\bar\partial_B}}:\mathrm{Hom}(\Omega^k_{\mathbb{C}},\Omega^k_{\mathbb{C}})\otimes S^2(T^*S)$ maps any non-zero $\omega\in T^*S$ to an automorphism on $\Omega^k_{\mathbb{C}}$. When calculating the symbol one essentially keeps only those terms of $\Delta_{\bar\partial_B}$ that are of highest order in derivatives. In the context of the tangential Cauchy-Riemann operator, this means that ∂_B and $\bar\partial_B$ can be taken to be anticommuting and that the overall result is essentially the same as for the symbol of the Dolbeault Laplacian on a Kähler manifold, provided one substitutes $\partial_{z^i}\mapsto\partial_{z^i} - \eta(\partial_{z^i})\pounds_\xi$. Therefore, $\sigma_{\Delta_{\bar\partial_B}}(\xi) = 0$ and $\Delta_{\bar\partial_B}$ is not elliptic.

An obvious problem of interest is the extension of the results presented here beyond the Sasaki-Einstein case. As long as there is a dual SCFT, there is a unitarity bound meaning that there should be some equivalent of (1) or at least (13). A starting point might be given by partially complex geometry, see e.g. the lecture notes [17].

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[3] In the remainder of this discussion, the term *holomorphic* is meant in respect to the tangential Cauchy-Riemann operator $\bar\partial_B$.

	L_η	Λ_η	L	Λ	∂_B^*	$\bar\partial_B^*$	$\bar\partial_B$
∂_B	$\{\partial_B, L_\eta\} = L$	$\{\partial_B, \Lambda_\eta\} = 0$	$[\partial_B, L] = 0$	$[\partial_B, \Lambda]$	Δ_{∂_B}	$\{\partial_B, \bar\partial_B^*\}$	$\{\partial_B, \bar\partial_B\} = -2L\pounds_\xi$
$\bar\partial_B$	$\{\bar\partial_B, L_\eta\} = L$	$\{\bar\partial_B, \Lambda_\eta\} = 0$	$[\bar\partial_B, L] = 0$	$[\bar\partial_B, \Lambda]$	$\{\bar\partial_B, \partial_B^*\}$	$\Delta_{\bar\partial_B}$	
$\bar\partial_B^*$	$\{\bar\partial_B^*, L_\eta\} = 0$	$\{\bar\partial_B^*, \Lambda_\eta\} = \Lambda$	$[\bar\partial_B^*, L]$	$[\bar\partial_B^*, \Lambda] = 0$	$\{\bar\partial_B^*, \partial_B^*\} = 2\Lambda\pounds_\xi$		
∂_B^*	$\{\partial_B^*, L_\eta\} = 0$	$\{\partial_B^*, \Lambda_\eta\} = \Lambda$	$[\partial_B^*, L]$	$[\partial_B^*, \Lambda] = 0$			
Λ	$[\Lambda, L_\eta] = 0$	$[\Lambda, \Lambda_\eta] = 0$	$[\Lambda, L] = (n - d^0)$				
L	$[L, L_\eta] = 0$	$[L, \Lambda_\eta] = 0$					
Λ_η	$\{\Lambda_\eta, L_\eta\} = 1$						

TABLE I. The Kähler-like identities

Appendix A: The identities

Table I lists the various (anti-) commutators. The more involved ones that do not fit in the table are listed in equation (A1).

$$\begin{aligned}
[\partial_B, \Lambda] &= -\imath\bar\partial_B^* + \imath L_\eta\Lambda - (n - d^0)\Lambda_\eta \\
[\bar\partial_B, \Lambda] &= \imath\partial_B^* - \imath L_\eta\Lambda - (n - d^0)\Lambda_\eta, \\
[\partial_B^*, L] &= -\imath\bar\partial_B + \imath\Lambda_\eta L - (d^0 - n)L_\eta, \\
[\bar\partial_B^*, L] &= \imath\partial_B - \imath\Lambda_\eta L - (d^0 - n)L_\eta, \\
\{\partial_B, \bar\partial_B^*\} &= \imath(L_\eta\bar\partial_B^* + \partial_B\Lambda_\eta) + (n - d^0)L_\eta\Lambda_\eta + L\Lambda, \\
\{\bar\partial_B, \partial_B^*\} &= -\imath(L_\eta\partial_B^* + \bar\partial_B\Lambda_\eta) + (n - d^0)L_\eta\Lambda_\eta + L\Lambda.
\end{aligned} \tag{A1}$$

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