Conformal Quantum Mechanics and Sine-Square Deformation

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Abstract

We revisit conformal quantum mechanics (CQM) from the perspective of sine-square deformation (SSD) and the entanglement Hamiltonian. The operators that correspond to the SSD and the entanglement Hamiltonian are identified. Thus, the nature of SSD and entanglement can be discussed in a much simpler CQM setting than higher dimensional field theories.

In [1,2], it was shown that sine-square deformation (SSD) [3] for two-dimensional (2d) conformal field theory (CFT) [4] can be understood by introducing a new quantization scheme called "dipolar quantization."[1)] The basic idea was generalized in Ref. [12] to incorporate the entanglement Hamiltonian and other interesting deformations of 2d CFT. In this Letter, we examine whether the idea of dipolar quantization is applicable to the one-dimensional (1d) case, which is called conformal quantum mechanics (CQM). CQM was first studied in the seminal paper by de Alfaro, Fubini, and Furlan [13].

To put the problem in perspective, let us consider a scalar field $\phi(x)$ on general d-dimensional flat spacetime $x^\mu(\mu = 0, \ldots, d-1)$ following the argument presented in Ref. [13]. Suppose $\phi(x)$ transforms under the scale transformation $x_\mu \to x'_\mu = \lambda x_\mu$ as

$$\phi(x^\mu) \to \phi'(x'^\mu) = \phi'(\lambda x^\mu) = \lambda^{-\frac{2d}{2}}\phi(x^\mu) \tag{1}$$

A simple invariant action for $\phi(x)$ can be obtained as

$$S = \int \prod_\mu dx^\mu \frac{1}{2}\left(\partial_\nu\phi\partial^\nu\phi - g\phi^{\frac{2d}{d-2}}\right), \tag{2}$$

where g is the dimensionless coupling constant. Because scale invariance implies conformal invariance in most cases [14], this action provides a good starting point.

1) See Refs. [5,6] for earlier studies on SSD and Refs. [9–11] for more recent studies. References [7,8] study SSD in the context of string theory and conformal field theory.

In addition, Eq. (2) suggests the difficulty in the Lagrangian formalism for $d = 2$ case, in which the energy momentum tensor is taken as the basis of the theory rather than the lagrangian.

The case of interest here is $d = 1$, which has the following Lagrangian:

$$L = \frac{1}{2}(\dot{q}(t))^2 - \frac{g}{2}\frac{1}{q(t)^2}, \tag{3}$$

where t is the 1d "spacetime" coordinate. We also changed the notation of the "field" from ϕ to $q(t)$ because we are now dealing with a quantum mechanical system. We can then show that the Lagrangian (3) possesses the following symmetry:

$$t \rightarrow t' = \frac{at+b}{ct+d}, \quad ad - bc = 1, \tag{4}$$

$$q(t) \rightarrow q'(t') = \frac{1}{ct+d}q(t), \tag{5}$$

which is a larger symmetry than scale invariance and translational invariance combined. In fact, it is 1d conformal symmetry, as we anticipated.

The transformation (4) for t can be conveniently decomposed into the following three components:
Translation $a = d = 1$ and $c = 0$ lead to

$$t \rightarrow t + b. \tag{6}$$

Dilatation $a = 1/d$ and $b = c = 0$ lead to

$$t \rightarrow a^2 t. \tag{7}$$

Special Conformal Transformation(SCT) $a = d = 1$ and $b = 0$ lead to

$$t \rightarrow \frac{t}{ct+1}. \tag{8}$$

The infinitesimal version of transformations (6) - (8) of the above three can be represented in terms of the differential operators as follows.

$$(Time)Translation \ \frac{d}{dt} \equiv P_0, \tag{9}$$

$$Dilatation \ t\frac{d}{dt} \equiv D, \tag{10}$$

$$SCT \ t^2\frac{d}{dt} \equiv K_0. \tag{11}$$

These operators form a closed algebra,

$$[D, K_0] = K_0, \ [D, P_0] = -P_0, \ [P_0, K_0] = 2D, \tag{12}$$

which is readily isomorphic to $sl(2,\mathbb{R})$ algebra or, equivalently, the subalgebra formed by the three Virasoro generators $L_1, L_0, andL_{-1}$:

$$[L_0, L_{-1}] = L_{-1},\ [L_0, L_1] = -L_1,\ [L_1, L_{-1}] = 2L_0. \tag{13}$$

The time-translation generator P_0 should be identified with the Hamiltonian

$$H = \frac{1}{2}p(t)^2 + \frac{g}{2}\frac{1}{q(t)^2}, \tag{14}$$

where p is the canonical momentum. Equation (14) was directly derived from Lagrangian (3). Using the symplectic structure, the rest of the generators may be expressed in terms of q and p:

$$K_0 = \frac{1}{2}q(t)^2, \tag{15}$$

$$D = -\frac{1}{4}\left(p(t)q(t) + q(t)p(t)\right). \tag{16}$$

In Eq. (16) we employed symmetrization in anticipation of quantization.

In Ref. [13], de Alfaro, Fubini, and Furlan introduced the new operator

$$R \equiv \frac{1}{2}\left(aP_0 + \frac{1}{a}K_0\right), \tag{17}$$

where a is a constant with the dimensions of time, along with two other operators. Then, R was proposed to supersede H as the time-translation operator, or the Hamiltonian.

The distinction between the operator R and the original Hamiltonian $H = P_0$ is best clarified from the symmetry viewpoint [2,13]. First, the (quadratic) Casimir invariant for $sl(2,\mathbb{R})$ algebra is

$$C_{(2)} = \frac{1}{2}L_{-1}L_1 + \frac{1}{2}L_1L_{-1} - (L_0)^2 = \frac{1}{2}K_0P_0 + \frac{1}{2}P_0K_0 - D^2. \tag{18}$$

Therefore, for any adjoint action of $sl(2,\mathbb{R})$ algebra on the linear combination of the generators,

$$x^{(0)}L_0 + x^{(1)}L_1 + x^{(-1)}L_{-1} \longrightarrow x'^{(0)}L_0 + x'^{(1)}L_1 + x'^{(-1)}L_{-1}, \tag{19}$$

the following combination remainsunchanged [2]:

$$2x^{(1)}x^{(-1)} + 2x^{(-1)}x^{(1)} - \left(x^{(0)}\right)^2 = 4x'^{(1)}x'^{(-1)} - \left(x'^{(0)}\right)^2 \equiv c^{(2)} \tag{20}$$

[2] Note that the numerical coefficients of the quadratic form in Eqs. (18) and (20) are components of matrices that are inverse of each other.

In terms of the coefficients $x^{(0)}, x^{(1)}, and x^{(-1)}$, the operator R is expressed as

$$R : x^{(0)} = 0, x^{(1)} = \frac{a}{2}, x^{(-1)} = \frac{1}{2a}, \tag{21}$$

and the expression for the original Hamiltonian H (or P_0) is

$$H : x^{(0)} = 0, x^{(1)} = 1, x^{(-1)} = 0. \tag{22}$$

Putting these coefficients into $c^{(2)}$ defined in Eq. (20), we immediately find

$$c^{(2)} = 1 \quad \text{for } R, \tag{23}$$

and

$$c^{(2)} = 0 \quad \text{for } H. \tag{24}$$

These results imply that one cannot connect R and H by any adjoint action of $sl(2,\mathbb{R})$, nor by its exponentiation, $SL(2,\mathbb{R})$. In this sense, operators R and H are disconnected.

Now, note the absence of constant a in expression (23), which infers that a can be changed numerically by an adjoint action of $sl(2,\mathbb{R})$ or $SL(2,\mathbb{R})$ action on the operator R. In fact, an infinitesimal change in $a \to a(1-\epsilon)$ evokes the commutation with D as

$$\begin{aligned} R \xrightarrow{a\to a(1-\epsilon)} \frac{1}{2}\left(a(1-\epsilon)P_0 + \frac{1}{a(1-\epsilon)}K_0\right) &= R + \frac{1}{2}\left(-aP_0 + \frac{K_0}{a}\right)\epsilon \\ &= R + [D, R]\epsilon. \end{aligned} \tag{25}$$

Thus, different values of a in R are connected by the action of D. Two other actions can be applied to R, namely, P_0 and K_0, which would produce terms corresponding to D and yield a nonzero $x^{(0)}$ coefficient. Hereinafter, we assume a to be unity for the sake of simplicity.

We then ask if any class of operators is connected to H by the action of $SL(2,\mathbb{R})$. Apparently, the answer is affirmative because the following operator $H^{(a,b)}$

$$H^{(a,b)} : x^{(0)} = \pm 2\sqrt{ab}, x^{(1)} = a, x^{(-1)} = b, \quad \text{for } \; ab \geq 0, \tag{26}$$

yields $c^{(2)} = 0$ as does H, which can be written in the above notation as

$$H = H^{(1,0)}. \tag{27}$$

$H^{(a,b)}$ is explicitly written as

$$H^{(a,b)} = aP_0 + bK_0 \pm 2\sqrt{ab}D, \tag{28}$$

or, in terms of the canonical variables,

$$H^{(a,b)} = \frac{a}{2}p(t)^2 + \frac{ag}{2}\frac{1}{q(t)^2} + \frac{b}{2}q(t)^2 - \sqrt{ab}p(t)q(t), \tag{29}$$

where, without loss of generality, we have chosen one of the double signs that appeared in Eq. (26).

The transformation between $H^{(1,0)}$ and $H^{(a,b)}$ can be interpreted in terms of classical mechanics because we have designed the system so that it accommodates conformal symmetry. In fact, the transformation can be achieved by changing the canonical coordinates as follows:

$$\begin{cases} q(t) \to \frac{1}{\sqrt{a}}Q(t) \\ p(t) \to \sqrt{a}P(t) - \sqrt{b}Q(t) \end{cases}, \tag{30}$$

where $P(t)$ and $Q(t)$ are the new canonical coordinates. The generating function of the above canonical transformation is

$$W = \sqrt{a}q(t)P(t) - \frac{\sqrt{ab}}{2}q^2(t). \tag{31}$$

Another class of generators yields negative $c^{(2)}$, the simplest of which is

$$\bar{R} \equiv H - K = \frac{1}{2}p(t)^2 + \frac{1}{2}\frac{g}{q(t)^2} - \frac{1}{2}q(t)^2. \tag{32}$$

For $\bar{R}$, the coefficients are

$$\bar{R} : x^{(0)} = 0, x^{(1)} = 1, x^{(-1)} = -1, \tag{33}$$

which yield $c^{(2)} = -1$. [3)]

Now, each distinct class of $c^{(2)}$ can be conveniently represented by the following combination of coefficients:

$$x^{(0)} = 0, x^{(1)} = 1, x^{(-1)} = \frac{c^{(2)}}{4}. \tag{34}$$

The corresponding generator is

$$H + \frac{c^{(2)}}{4}K_0 = \frac{1}{2}p^2 + \frac{g}{2}\frac{1}{q^2} + \frac{c^{(2)}}{8}q^2. \tag{35}$$

Because the above generator resembles the ordinary Hamiltonian, it is clarifying to draw the graph of the potential $V(q) = \frac{g}{2}\frac{1}{q^2} + \frac{c^{(2)}}{8}q^2$ for each case. Figure 1

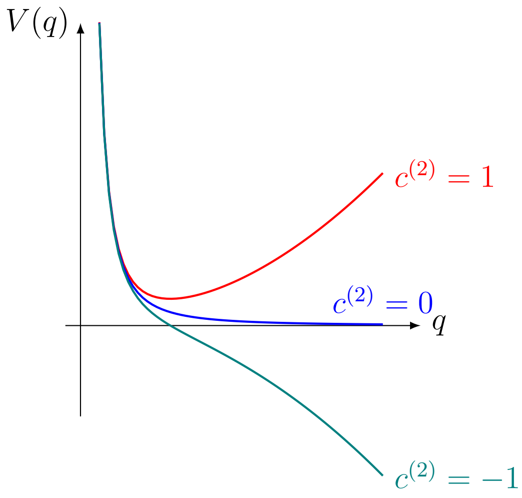

Figure 1: Potential $V(q)$ for $c^{(2)} = 1, 0, and - 1$.

shows the potential for the cases where $c^{(2)}$ equals 1, 0, and -1, respectively. In the following, we investigate each case.

Reference [13] observed that the invariance of the Casimir invariant (18) is apparent from expressions (14) - (16), if one imposes the commutation relation over q and p as $[q, p] = i\mathbb{I}$:

$$\frac{1}{2}HK_0 + \frac{1}{2}K_0H - D^2 = \left(\frac{g}{4} - \frac{3}{16}\right)\mathbb{I}, \tag{36}$$

where $\mathbb{I}$ is the identity operator of the (enveloping) algebra in question [4)]. Without fear of confusion, we also denote the parameter $\left(\frac{g}{4} - \frac{3}{16}\right)$ as $C_{(2)}$. Using the notation $C_{(2)}$, one obtains

$$L_{\pm 1}L_{\mp 1} = L_0^2 \pm L_0 - C_{(2)}\mathbb{I} \tag{37}$$

which turns out to be useful for finding the eigenvalues of L_0.

Suppose a normalized eigenstate vector $|E\rangle$ exists such that

$$L_0|E\rangle = E|E\rangle, \quad \langle E|E\rangle = 1. \tag{38}$$

[3)] $\bar{R}$ corresponds to $-S$ in the notation of Ref. [13].

[4)] As noted in Ref. [13], the case $g = 0$ yields particularly simple representations by the creation and annihilation operators $[a, a^\dagger] = 1$, which are called singleton representations. Despite the ostensible lack of enough structure to accommodate the symmetry, this is an example of a spectrum generating algebra, and the symmetry algebra is represented by the transitions between the different energy states [15, 16].

It is then straightforward to show that one can construct eigenstates witheigen-values $E \pm 1$ by multiplying by $L_{\mp 1}$ because

$$L_0 \left(L_{\mp 1} |E\rangle \right) = \left(L_{\mp 1} L_0 \right) |E\rangle + L_{\mp 1} |E\rangle = (E \pm 1) L_{\mp 1} |E\rangle. \tag{39}$$

We would like to normalize the eigenstates obtained above,

$$L_{\mp 1} |E\rangle \equiv c^{\pm}(E) |E \pm 1\rangle, \tag{40}$$

so that

$$\langle E \pm 1 | E \pm 1 \rangle = 1. \tag{41}$$

The normalization factor $c^{\pm}$ can be calculated using Eq. (37), which yields

$$|c^{\pm}(E)|^2 = \langle E | L_{\pm 1} L_{\mp 1} | E \rangle = \langle E | L_0^2 \pm L_0 - C_{(2)} \mathbb{I} | E \rangle = E^2 \pm E - C_{(2)} \geq 0. \tag{42}$$

This condition of positivity can be clearer if we introduce a common notation for the Casimir invariant $C_{(2)} = j(j-1)$ (we assume $j \geq 0$):

$$|c^{\pm}(E)|^2 = E(E \pm 1) - j(j-1) \geq 0. \tag{43}$$

We thus conclude that $E \geq j$ or $E \leq -j$, and from physical considerations, we prefer positive E. Finally, as the eigenvalues of L_0, we obtain

$$E = n + j, \tag{44}$$

where $n = 0, 1, 2, 3, \ldots$ [5]. Because R can be identified with L_0, we obtain the system with a discrete spectrum using R as the Hamiltonian in stead of the original H. This is fairly evident from Fig. 1 because R corresponds to the case $c^{(2)} = 1$, where the range of motion is apparently limited to a finite region.

Conversely, the case $c^{(2)} = 0$ does not exhibit discrete energy states because the motion of the particle is not confined by the potential. Instead, it has a continuous spectrum as discussed in detail in Ref. [13]. This emergence of the continuous spectrum compelled the authors of Ref. [13] to propose R as the Hamiltonian of CQM instead of the original H, which corresponds to the case $c^{(2)} = 0$.

However, we prefer to propose another interpretation of H here: In light of SSD, we do not have to reject an operator just because it leads a continuous spectrum. In fact, this is the signature of SSD systems. Therefore, we propose to regard H as the SSD Hamiltonian. If we accept this interpretation, the relation between radial quantization [18] and SSD in 2d conformal field theories [1, 2] naturally parallels that between R and H. This interpretation offers a nice intuition on somewhat

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[5] One might consider an extension of $sl(2, \mathbb{R})$ to the full Virasoro algebra on these eigenstates. See Ref. [17] for a related discussion.

mysterious nature of the continuous spectrum of SSD: it stems from the runaway potential in the CQM case.

Next, we turn our attention to the case $c^{(2)} = -1$. Since the potential for this case is unbounded below, the system is unstable, no meaningful physical interpretation is apparent. However, the $sl(2, \mathbb{R})$ symmetry of the system enables the following analysis.

First, the generators H, K_0, D, and R allows another non-trivial identification with the Virasoro subalgebra:

$$L'_0 = \frac{1}{2i}(H - K_0) = \frac{1}{2i}\bar{R}, \tag{45}$$

$$L'_{-1} = -\frac{1}{2}(H + K_0) - D = -\frac{1}{2}R - D, \tag{46}$$

$$L'_1 = \frac{1}{2}(H + K_0) - D = \frac{1}{2}R - D. \tag{47}$$

The set of operators above satisfies the same commutation relations given in Eq. (13). Since the algebraic structure is the same, the eigenvalues for the operator L'_0 should be the same. However, the "Hamiltonian" in question is $\bar{R}$, not L'_0. The difference between R and L_0 is the multiplication of the the imaginary unit i. Thus we find that the spectrum of $\bar{R}$ is $2i$ times that of R.

Next, what can we make of a "Hamiltonian" with pure imaginary eigenvalues? Although imaginary eigenvalues appear unphysical, all these eigenvalues take the form $2iE_n$, where E_n represent the eigenvalues of the "physical" Hamiltonian R, as explicitly shown in Eq. (44). If we take $t \to \beta/2$, the "time" translation becomes

$$\exp(it\bar{R}) = \sum |n\rangle e^{-\beta E_n} \langle n|. \tag{48}$$

It is clear that the above operator corresponds to the thermal density matrix operator

$$\rho \equiv \frac{\exp(-\beta R)}{\mathrm{Tr}\left[\exp(-\beta R)\right]}. \tag{49}$$

The relation between the density matrix operator ρ and $\bar{R}$,

$$\bar{R} \sim -\frac{1}{\beta} \ln \rho \tag{50}$$

is reminiscent of the entanglement Hamiltonian [19–26]. Therefore, we infer that this case corresponds to the entanglement Hamiltonian. Because the system has only one degree of freedom and there is no other degree of freedom to integrate out, we naturally obtain the expression for the entire density matrix from the corresponding entanglement Hamiltonian for $d = 1$.

At this point, it would be insightful to contemplate the action of $sl(2,\mathbb{R})$. The $sl(2,\mathbb{R})$ algebra is also the Lie algebra of the projective special linear group $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm\}$ which is apparently a subgroup of $SL(2,\mathbb{R})$. The relation between $PSL(2,\mathbb{R})$ and $SL(2,\mathbb{R})$ is reminiscent of the relation between $SO(3)$ and $SU(2)$. $PSL(2,\mathbb{R})$ naturally acts on the hyperbolic plane $\mathbb{H}^2$, which is the upper half of the complex plane $\{z \in \mathbb{C}; \mathrm{Im}z > 0\}$ with the Poincaré metric

$$ds^2 = \frac{|dz|^2}{(\mathrm{Im}z)^2}, \tag{51}$$

or the Poincaré disk with the metric

$$ds^2 = \frac{|dz|^2}{(1-|z|^2)^2}. \tag{52}$$

The action of $PSL(2,\mathbb{R})$ on $\mathbb{H}^2$ gives the following automorphism:

$$z \mapsto \frac{az+b}{cz+d}, \tag{53}$$

where $a,b,c,d \in \mathbb{R}$ and $ad-bc \neq 0$. The action of $PSL(2,\mathbb{R})$ on the Poincaré disk that corresponds to R, H, and $\bar{R}$ respectively, is depicted in Fig. 2

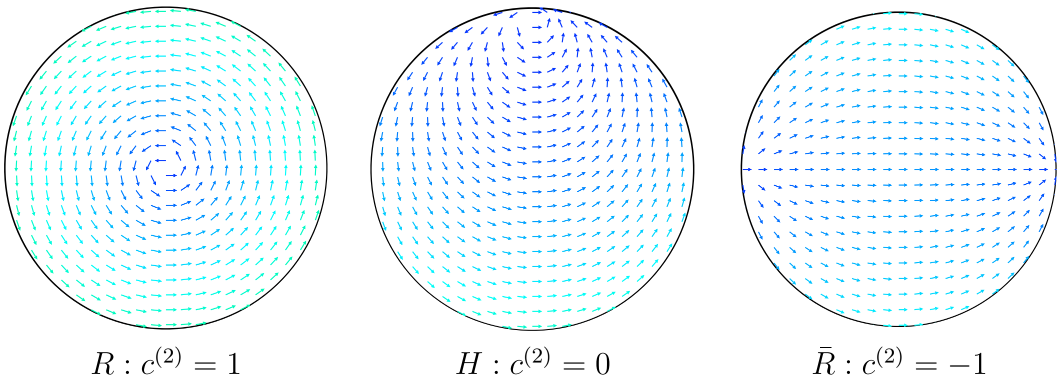

Figure 2: Time translation on the Poincaré disk. On the boundary of the disk, "time flow" is uniform without fixed point for R or $c^{(2)} = 1$ case, while it is limited to the finite region bounded by the two fixed points for $\bar{R}$ or $c^{(2)} = -1$ case. H or $c^{(2)} = 0$ case exhibits the marginal behavior, and it has one fixed points at infinity.

Note, in particular, that the Möbius transformation is similar to the transformation above, except that it forms a complex Lie group and is isomorphic to the automorphism of the Riemann sphere $\mathrm{Aut}(\widehat{\mathbb{C}})$ rather than to the automorphism of the half plane. The observation here would also be useful in the study of SSD for the case of open strings, where the setup of the upper half plane is natural.

In summary, we find the same structure in CQM as observed in 2d CFT where the choice of the Hamiltonian leads to radial quantization, the dipolar quantization or SSD, and the entanglement Hamiltonian, respectively. We identify the respective Hamiltonians in CQM using $sl(2,\mathbb{R})$ symmetry. The findings here will offer a simpler setup for the study of SSD and the entanglement Hamiltonians. It would be also interesting to investigate further in the context of the conformal boot strap approach [27] or the recent discussion of the CQM correlation function [28].
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