

Notes on Enhancement of Flavor Symmetry and 5d Superconformal Index

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Abstract

Five-dimensional minimal supersymmetric gauge theories are of course non-renormalizable but some of them are completely defined quantum-mechanically because of the existence of the UV fixed points. The UV fixed point theory of such a 5d gauge theory is believed to have an enlarged flavor symmetry. $SU(2)$ gauge theory with $N_f = 0, 1, \dots, 7$ flavors for instance leads to UV superconformal fixed point with E_{N_f+1} flavor symmetry. Actually it is not easy to check this conjecture because the UV theory is strongly-coupled, however, computation of certain SUSY protected quantities provides strong evidence for the enhancement of flavor symmetry. We study the superconformal index for $SU(2)$ gauge theory with $N_f = 0, 1$ flavors in details, and we provide an all-order evidence of the enhancement by studying combinatorial structure of the superconformal indexes of these theories. We also give a nontrivial evidence that the local \mathbb{F}_2 geometry leads to $SU(2)$ pure Yang-Mills theory with zero discrete theta term.

1 Introduction

Perturbative renormalizability has been a criterion for the predictable quantum field theory. Needless to say, this is because the renormalization removes ultraviolet (UV) divergences of a Feynman diagram and leads to a meaningful finite value of a physical quantity. While an effective theory is permitted to include non-renormalizable interactions, of course, this criterion must be satisfied by a fundamental theory without any cut-off scale and excludes many models of the quantum field theory. The renormalizable theories, however, do not exhaust all possibilities.

Actually a quantum theory endowed with a UV fixed point is well defined and valid at the whole energy scale. This possibility is known as the Weinberg asymptotic safety scenario [1], which perhaps rescues the non-renormalizability of the perturbative quantum gravity. This scenario is also very attractive because a renormalizable but asymptotically non-free theory such as pure QED involves the Landau pole and the convergence radius of the perturbation becomes zero according to popular apprehension. With assuming the existence of the UV fixed point, we can avoid such a theoretical inconsistency included in perturbative quantum field theory.

UV fixed point is very important notion of the quantum field theory, but it is very hard in general to determine whether a theory has a UV fixed point or not. Five-dimensional minimal supersymmetric gauge theories are typical and attractive exceptions to this difficulty. Perturbative five-dimensional gauge theories are non-renormalizable, but Seiberg [2] showed that perturbative description breaks down at high energy but some of such theories flows up to a strongly coupling, non-Gaussian, UV fixed point. $SU(2)$ gauge theory with $N_f = 0, 1, \dots, 7$ fundamental flavors provides a concrete example [3, 4]. The flavor symmetry of this gauge theory is $SO(2N_f) \times U(1)_I$, where $U(1)_I$ is associated with the instanton number. The UV fixed point is described by a strongly coupled conformal field theory. At this fixed point, the flavor symmetry is expected to enhance to the larger group E_{N_f+1} : $E_1 = SU(2)$, $E_2 = SU(2) \times U(1)$, $E_3 = SU(3) \times SU(2)$, $E_4 = SU(5)$, $E_5 = SO(10)$, and $E_{6,7,8}$ are the usual exceptional Lie groups.

This enhancement of the flavor symmetries was conjectured by employing superstring theory [2], and so far it was not easy to show this enhancement based only on field theory arguments. This is because the UV fixed point theories in question are strongly coupled, and it has prevented us from verifying this conjecture directly. Fortunately,

with recent progress in the theories of the localization and the superconformal index, we can discuss the strongly-coupled fixed point theories quantitatively by evaluating the protected indexes of these theories. This direction was opened by Kim, Kim and Lee [5], and their result was reformulated and extended by the work of Iqbal and Vafa [6]. In this paper we study the detailed structure of their result on the five-dimensional superconformal index, and we provide a justification of the enhancement of the flavor symmetry for $N_f = 0, 1$.

The structure of the paper is as follows. We give a brief review on background materials in section 2. In section 3 we study some superconformal field theories associated with 5-brane web constructions by computing the superconformal indexes of them. We conclude in section 4. In appendix A, we set some conventions, deriving useful formulas.

2 5d Superconformal Index and Nekrasov formula

In this paper we will discuss five-dimensional $\mathcal{N} = 1$ superconformal field theories, and these theories enjoy the superconformal symmetry named $F(4)$. The bosonic part of $F(4)$ is $SO(2, 5) \times SU(2)_R$. The bosonic conformal group contains $SO(2) \times SO(5)$, and this $SO(5)$ is the five-dimensional Lorentz group. The $SO(2)$ charge gives the dimension ϵ_0 . Let $A = 1, 2$ be the $SU(2)_R$ index and $m = 1, 2, 3, 4$ be the spinor index of $SO(5)$. Let Q_m^A and S_A^m be the super and superconformal charges.

To define the superconformal index, let us pick up the generators $\mathcal{Q} = Q_2^1$ and $\mathcal{S} = \mathcal{Q}^\dagger = S_1^2$. Then the above commutation relation gives

$$\Delta = \{\mathcal{Q}, \mathcal{S}\} = \epsilon_0 - 2j_1 - 3R \geq 0. \quad (2.1)$$

The $SU(2)_L \times SU(2)_R \subset Sp(4)$ Cartans are $j_{1,2}$, which are given by the Cartans of $SO(5)$ in the orthogonal basis $h_{1,2}$

$$j_1 = \frac{h_1 + h_2}{2}, \quad j_2 = \frac{h_1 - h_2}{2}. \quad (2.2)$$

The Cartan of $SU(2)_R$ is denoted by R . The states annihilated by \mathcal{Q} and \mathcal{S} are then the $\frac{1}{8}$ BPS states. Since they saturate the positivity bound, these states have the dimension $\epsilon_0 = 2j_1 + 3R$. To count these minimally BPS states we can use the Witten index

$$I = \text{Tr} \mathcal{H}_{\frac{1}{8} \text{ BPS}} (-1)^F = N_{\text{bosons}} - N_{\text{fermions}}. \quad (2.3)$$

This index counts the BPS states which are annihilated by \mathcal{Q} and $\mathcal{Q}^\dagger = \mathcal{S}$. Even in the case where I is finite, the numbers N_{bosons} and N_{fermions} are infinite in general. To make the expression meaningful, we need to introduce regulators which correspond to the Cartan generators commuting with \mathcal{Q} , \mathcal{S} , and each other. In our case these commuting generators are

$$\Delta, \quad j_1 + R, \quad j_2, \quad H_f, \quad J = * \text{tr}(F \wedge F), \quad (2.4)$$

where H_f are the Cartans of flavor symmetry, and J is the conserving instanton current. We then arrive at the following definition of the so-called superconformal index:

$$I_{5\text{d}}(u, z_f, t, q) = \text{Tr}_{\mathcal{H}_{\frac{1}{8}}^{\text{BPS}}} (-1)^F e^{-\beta\{\mathcal{Q}, \mathcal{S}\}} x^{2(j_1+R)} y^{2j_2} \prod_f z_f^{H_f} u^k. \quad (2.5)$$

k counts the instanton charge. The fugacities x and y count the $j_{1,2}$ charges, and we can introduce $SO(5)$ Cartan fugacities t and q through the relation

$$x = \sqrt{\frac{q}{t}}, \quad y = \sqrt{qt}. \quad (2.6)$$

We can recast the index into the Euclidean path integral over fields with twisted boundary conditions

$$I_{5\text{d}}(z_f, t, q) = \int_{\text{twisted b.c.}} \mathcal{D}\Phi e^{-S_{S^1 \times S^4}[\Phi]}. \quad (2.7)$$

Kim, Kim, and Lee [5] performed the path integration by employing the localization method¹ *a la* Pestun, and they found the following expression of the superconformal index:

$$I_{5\text{d}}(z_f, t, q) = \oint \prod_\alpha [da_\alpha] |Z_{5\text{d}}^{\text{Nek.}}(Q_\alpha, z_f; t, q)|^2, \quad (2.8)$$

where $[da_\alpha]$ is the Haar measure for loop variables $Q_\alpha = e^{ia_\alpha}$ and we introduce new fugacities as $q = xy$ and $t = \frac{y}{x}$. Here $Z_{5\text{d}}^{\text{Nek.}}$ is the 5d Nekrasov partition function [9], and the integral is taken over $|Q_\alpha| = 1$ for the loop variables $Q_\alpha = e^{ia_\alpha}$. The conjugation $\bar{*}$ is defined by $\overline{f(Q_\alpha, z_f; t, q)} = f(Q_\alpha^{-1}, z_f^{-1}; t, q)$ for generic function. The two integrands $Z_{5\text{d}}^{\text{Nek.}}$ and $\overline{Z_{5\text{d}}^{\text{Nek.}}}$ are the contributions from the north and south poles of the sphere S^4 , where the fixed points of the localization computation.

¹The localization on $S^1 \times S^4$ was also formulated and computed in [7, 8].

Iqbal and Vafa [6] gave a string-theoretical reformulation of the above formula. Let us recall the idea of the geometric engineering: a 5d $\mathcal{N} = 1$ gauge theory arises from certain Calabi-Yau compactification of M theory. As a consequence of the stringy realization, the partition function of the 5d theory is exactly equal to the refined topological string partition function Z_{CY} of the corresponding CY. This topological string partition function is defined by the refinement [10, 11] of the topological vertex formalism [12]-[24], which is the large- N dual to the refined Chern-Simons theory through the geometric transition [19, 20]. See [10]-[23] for details of this formalism. The 5d index is then the following loop integral

$$I_{5\text{d}}(z_f, t, q) = \oint \prod_{\alpha} \frac{dQ_{\alpha}}{Q_{\alpha}} Z_{\text{CY}}(Q_i; t, q) Z_{\text{CY}}(Q_i^{-1}; t^{-1}, q^{-1}), \quad (2.9)$$

where a loop variable Q_{α} is assigned for each loop in the web diagram, and $Q = e^{-t}$ is the exponential of the Kähler parameter t . In fact, in this formula which was derived by Iqbal and Vafa from the perspective of topological string theory, the complex conjugation acts not only on Q_i but also on t and q , and so the expression is slightly different from that of Kim, Kim and Lee. In [6] the authors however showed that usual topological string partition functions satisfy the relation $Z_{\text{CY}}(Q_i^{-1}; t^{-1}, q^{-1}) = Z_{\text{CY}}(Q_i^{-1}; t, q)$, and then we can recover the expression derived by Kim, Kim and Lee. This is not always the case because when toric Calabi-Yau manifold contains a flat deformation direction of two cycle as the case of $\mathcal{O}(0) \oplus \mathcal{O}(-2) \rightarrow \mathbb{C}\mathbb{P}^2$, the massive spectrum does not transform as a representation of the Wigner little group [6, 27, 28]. We therefore need to remove the factor coming from the problematic states which do not form any full spin content of $SU(2)_L \times SU(2)_R$. The formalism to clear the ugly contribution away was recently proposed in [27, 28], and then we can obtain the renormalized partition function which enjoys the wanted invariance under the transformation $t \rightarrow t^{-1}$ and $q \rightarrow q^{-1}$.

The point is that Iqbal and Vafa proposed that this formula (2.9) also works for 5d theories without any gauge theory description nevertheless this Lagrangian description usually enables us to compute a Nekrasov partition function and the corresponding superconformal index exactly. This is because that the refined topological vertex formalism [11] extended by Iqbal and Kozcaz [23] provides the topological string partition function for any toric Calabi-Yau three-fold such as $\mathcal{O}(-3) \rightarrow \mathbb{P}^2$ which does not lead to any 5d gauge theory. The resulting partition function then provides the 5d superconformal

index via the Iqbal-Vafa proposal (2.9). In this way we can compute the 5d index only by using the web diagram of a theory.

Actually, complete refined topological string partition functions including the constant map contribution take the following form² [6]:

$$(M(t, q) M(q, t))^{\frac{\chi(\text{CY})}{4}} Z'_{\text{CY}}(Q_i; t, q), \quad (2.10)$$

where $M(t, q)$ is a two-parameter extension of the MacMahon function, and Z'_{CY} is the part that is given by the refined topological vertex. Since we study toric Calabi-Yau manifolds with single compact four-cycle in this paper, let the Euler number $\chi(\text{CY})$ be 2 in the following. The definition of this MacMahon function can be found in the next section. If the M-theory compactified on a Calabi-Yau manifold leads to a five-dimensional gauge theory, the corresponding refined topological vertex partition function can be decomposed in the following two parts; the perturbative and instanton contribution:

$$\begin{aligned} Z'_{\text{CY}}(Q_F, Q_B; t, q) &= \sum_{\vec{Y}} Q_B^{|\vec{Y}|} Z_{\vec{Y}}(Q_F; t, q) = Z_{\text{CY}}^{\text{pert.}}(Q_F; t, q) Z_{\text{CY}}^{\text{inst.}}(Q_F, Q_B; t, q), \\ Z_{\text{CY}}^{\text{pert.}}(Q_F; t, q) &= Z_{\vec{\emptyset}}(Q_F; t, q). \end{aligned} \quad (2.11)$$

The Nekrasov expression of the partition function $Z_{\text{CY}} = \sum_{\vec{Y}} Q_B^{|\vec{Y}|} Z_{\vec{Y}}$ naturally arises from the cut-and-glue process for toric diagrams in the topological vertex formalism. The full partition function is thus the following combination of the perturbative and instanton contributions:

$$Z_{\text{CY}}(Q_i; t, q) = \sqrt{M(t, q) M(q, t)} Z_{\text{CY}}^{\text{pert.}}(Q_i; t, q) Z_{\text{CY}}^{\text{inst.}}(Q_i; t, q). \quad (2.12)$$

This expression corresponds to the formula of Kim, Kim and Lee [5],

$$I_{5d}(z_f; t, q) = \int [da_\alpha] \text{PE}[f_{\text{mat.}}(x, y, Q_m, Q_{F\alpha}) + f_{\text{vec.}}(x, y, Q_{F\alpha})] \cdot |I^{\text{inst.}}|^2, \quad (2.13)$$

where the contributions from the two fixed points correspond to the instanton partition function $Z_{\text{CY}}^{\text{inst.}} = I^{\text{inst.}}$ and the remaining part gives $|\sqrt{M(t, q) M(q, t)} Z_{\text{CY}}^{\text{pert.}}|^2 dQ/Q$.

The Coulomb branch deformation is parametrized by Q_F , and masse deformations are Q_B and the Kähler parameters of the exceptional curves Q_{Ei} . Interpretation of these parameters is as follows [29, 30]: singular limit of a web diagram describes a superconformal

²We omit the classical contribution because it does not contribute to the superconformal index.

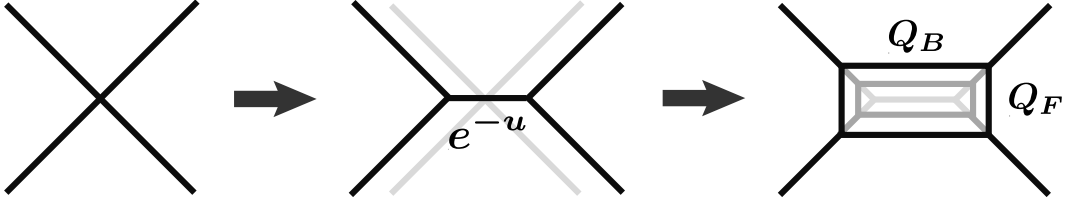


Figure 1: The first deformation which moves the external lines is the mass deformation of the theory E_1 . The second arrow denotes the breathing mode parametrized by Q_F .

point of the corresponding 5d theory. The left hand side of **Fig.1** illustrates the singular limit of the $SU(2)$ pure Yang-Mills theory. This figure describes the web diagram of the toric Calabi-Yau three-fold and the dual (p, q) 5-brane web. The deformation which moving the external lines of the web diagram is a mass deformation and parametrized by u and u_i 's. This global deformation moves infinitely massive branes, and it is associated with the Cartan of the global symmetry of a 5d theory. By giving VEV for the scalar component of the background vector multiplets for the global symmetry, this deformation is generated. This deformation thus changes the parameters of the theory. On the one hand, the breathing mode is a deformation of the web without moving the external lines. Only 5-branes with finite length are shifted under this deformation, and this variation is the change of the size of the loops inside the web diagram. We introduce the parameters Q_F for each loop to parametrize this deformation³. By turning on the scalar component of the vector multiplets of the local symmetry of a theory, we can generate this deformation. This is the deformation along the Coulomb branch of the theory.

In the following section, we provide explicit equations which relate the Kähler parameters with the Coulomb branch parameters and the fugacities.

3 Enhancement of Flavor Symmetry via 5d Index

In this section, we prove the enhancement of the flavor symmetry of $SU(2)$ gauge theories in the level of the 5d superconformal index. In the pioneer work [5] the enhancement was checked up to certain order of the power expansion of the index. We will provide an all order proof for some $SU(2)$ gauge theories.

³In this article we focus only on toric Calabi-Yau manifolds with single loop.

We introduce the fugacities x and y by the Ω -background through the relation

$$q = xy, \quad t = \frac{y}{x}, \quad (3.1)$$

and we expand a superconformal index as a positive power series in x . To do so, we have to introduce the following analytic continuation for combinatorial formulas. For instance, the refined MacMahon function

$$M(t, q) = \prod_{i,j=1}^{\infty} (1 - q^i t^{j-1})^{-1} = \prod_{i,j=1}^{\infty} (1 - x^{i-j+1} y^{i+j-1})^{-1}, \quad (3.2)$$

appears in the superconformal indexes. Since the naive geometric power series expansion of this expression contains infinitely many negative power of x , certain analytic continuation is needed to obtain the appropriate series expansion in $x \ll 1$. We employ the continuation in the sense of the relation $\sum_{n=0}^{\infty} t^n = -\sum_{n=0}^{\infty} t^{-n-1}$. Then the preferred expression of MacMahon function in region $|x| \ll 1$ becomes⁴

$$\begin{aligned} M(t, q) &= \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{(1 - q^n)(1 - t^n)} \right) \\ &= \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n t^{-n}}{(1 - q^n)(1 - t^{-n})} \right) \\ &= \prod_{i,j=1}^{\infty} (1 - q^i t^{-j}) = \prod_{i,j=1}^{\infty} (1 - x^{i+j} y^{i-j}). \end{aligned} \quad (3.3)$$

This expression immediately implies a positive-power expansion with respect to x , and therefore we employ such an continuation procedure to get the appropriate series expansion of a superconformal index.

The formula (2.9) then gives the following expression

$$I_{5d}(u; t, q) = M(t, q) M(q, t) \oint \prod_{\alpha} \frac{dQ_{\alpha}}{2\pi i Q_{\alpha}} Z_{\text{CY}}^{\text{t.v.}}(Q_i; t, q) Z_{\text{CY}}^{\text{t.v.}}(Q_i^{-1}; t, q), \quad (3.4)$$

where we use the identity $M(t^{-1}, q^{-1}) = M(q, t)$ to get this expression. In the following we investigate the detailed structure of this combinatorial expression in some concrete

⁴Since the function $M(t, q)$ has dense poles along $|t| = 1$, this deformation of the formula is not proper analytic continuation. We deal with a partition function as a formal power series in t and q with imposing the relation $\sum_{n \in \mathbb{Z}} t^n = \delta(t - 1)$. See [31] for rigorous treatment of such functions.

examples. This formula has the overall factor $M(t, q)M(q, t)$. We need a converging positive power series expansion in a variable $|x| \ll 1$, however analytic continuation to this region implies

$$\begin{aligned}
M(t, q)M(q, t) &= \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{t^n + q^n}{(1 - q^n)(1 - t^n)} \right) \\
&= \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(xy)^n + (y/x)^n}{(1 - (xy)^n)(1 - (y/x)^n)} \right) \\
&= \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{x^n ((y)^n + (1/y)^n)}{(1 - (xy)^n)(1 - (x/y)^n)} + 1 \right) \right), \tag{3.5}
\end{aligned}$$

and then the argument of the exponential function is clearly not converging. We therefore have to introduce the following modified factor by eliminating the diverging contribution $\sum_n 1/n$

$$\mathcal{M}(x, y)^2 = M(t, q) \tilde{M}(q, t) = \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} \frac{x^n ((y)^n + (1/y)^n)}{(1 - (xy)^n)(1 - (x/y)^n)} \right). \tag{3.6}$$

Actually the same regularization is also employed in the paper of Kim, Kim and Lee to derive the one loop partition function from the localization computation.

3.1 E_1 and \tilde{E}_1 theories

The $SU(2)$ gauge theories in 5d are special because there exist two theories distinguished by two topologically inequivalent configurations $\pi_4(SU(2)) = \mathbb{Z}_2$ [32]. This discrete group \mathbb{Z}_2 labels the allowed two value of the theta angle ϑ . We start with the UV fixed point theory for the pure Yang-Mills theory with vanishing theta angle. The local \mathbb{F}_0 is the canonical line bundle over the Hirzebruch surface $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, and this local Calabi-Yau threefold engineers this five-dimensional minimal-supersymmetric $SU(2)$ Yang-Mills theory. At the strong coupling, all the compact two cycles in the toric web diagram collapse and the diagram becomes singular as the left hand side of **Fig.1**. The deformation from the SCFT point is illustrated in **Fig.1**. The global mode u is then given by $Q_B = uQ_F$, and the local deformation is parametrized by $Q_F = e^{2ia}$.

E_1 theory and local \mathbb{F}_0 partition function

By turning on a relevant operator deformation associated with the background gauge field kinetic term for the Cartan sub-algebra of the flavor symmetry, E_1 SCFT flows down to 5d $SU(2)$ pure super Yang-Mills theory [2], which is engineered by Type IIA superstring theory on the local Hirzebruch surface \mathbb{F}_0 . A point is that we can compute the index of the UV SCFT by using this IR gauge theory since the superconformal index is RG-invariant quantity. The conventional topological string partition function [11], namely the Nekrasov partition function [9], which provides this superconformal index is

$$Z_{\mathbb{F}_0}(Q_B, Q_F; t, q) = \mathcal{M}(x, y) \sum_{\vec{R}} Q_B^{|\vec{R}|} q^{\|\vec{R}_2\|^2} t^{\|\vec{R}_1\|^2} \tilde{Z}_{R_1}(t, q) \tilde{Z}_{R_2}(q, t) \tilde{Z}_{R_1^t}(q, t) \tilde{Z}_{R_2^t}(t, q) \\ \times \prod_{i,j=1}^{\infty} \frac{1}{(1 - Q_F t^{i-1-R_{2,j}} q^{j-R_{1,i}}) (1 - Q_F t^{j-R_{2,i}} q^{i-1-R_{1,j}})}. \quad (3.7)$$

Here $\mathcal{M}(x, y)$ is the constant map contribution, and \tilde{Z}_R is a normalized Macdonald function in the principal specialization. See Appendix.A for the concrete definition of these function. The partition function can be factorized into the one-loop and instanton parts:

$$Z_{\mathbb{F}_0}(Q_B, Q_F; t, q) = \mathcal{M}(x, y) Z_{\text{pert}}^{N_f=0}(Q_F; t, q) Z_{\text{inst}}^{N_f=0}(Q_B, Q_F; t, q). \quad (3.8)$$

Here the combination $Z_{\text{pert}}^{N_f=0} Z_{\text{inst}}^{N_f=0}$ is the topological string partition function for the local \mathbb{F}_0 which is given by the refined topological vertex formalism. The perturbative and instanton part of the five-dimensional Nekrasov partition function take the following forms

$$Z_{\text{pert}}^{N_f=0}(Q_F; t, q) = \prod_{i,j=1}^{\infty} (1 - Q_F q^i t^{j-1})^{-1} (1 - Q_F q^{i-1} t^j)^{-1}, \quad (3.9)$$

$$Z_{\text{inst}}^{N_f=0}(u Q_F, Q_F; t, q) = \sum_{\vec{Y}} \left(u \frac{q}{t}\right)^{|\vec{Y}|} \prod_{\alpha, \beta=1}^2 \frac{1}{N_{Y_\alpha, Y_\beta}(Q_{\beta\alpha}; t, q)}, \quad (3.10)$$

where u is the instanton factor, and the combinatorial factors are

$$N_{R,Y}(Q; t, q) = \prod_{(i,j) \in R} (1 - Q t^{Y_j^t - i} q^{R_i - j + 1}) \prod_{(i,j) \in Y} (1 - Q t^{-R_j^t + i - 1} q^{-Y_i + j}), \quad (3.11)$$

$$Q_{21} = Q_F = Q_{12}^{-1}, \quad Q_{11} = Q_{22} = 1. \quad (3.12)$$

See appendix for details on the Nekrasov partition functions. Let us consider the instanton expansion

$$Z_{\text{inst}}(uQ_F, Q_F; t, q) = 1 + \sum_{n=1}^{\infty} u^n Z^{(n)}(Q_F; t, q). \quad (3.13)$$

Then the each contribution with fixed instanton number satisfies the following equality:

$$Z^{(n)}(Q_F^{-1}; t^{-1}, q^{-1}) = Z^{(n)}(Q_F; t, q) = Z^{(n)}(Q_F^{-1}; t, q) \quad (3.14)$$

We give a proof of this relation in Appendix.A. Then the integrant of the superconformal index becomes

$$|Z_{\text{inst}}^{N_f=0}(uQ_F, Q_F; t, q)|^2 = Z_{\text{inst}}^{N_f=0}((uQ_F)^{-1}, Q_F^{-1}; t, q) Z_{\text{inst}}^{N_f=0}(uQ_F, Q_F; t, q) \quad (3.15)$$

$$= \sum_{n=0}^{\infty} (u^n + u^{-n}) F^{(n)}(Q_F; x, y) \quad (3.16)$$

where the fugacities in [5] are given by $q = xy$ and $t = \frac{y}{x}$ and $F^{(n)}$ is

$$F^{(0)}(Q; x, y) = \frac{1}{2} \sum_{\ell=0}^{\infty} Z^{(\ell)}(Q; t, q)^2, \quad (3.17)$$

$$F^{(n)}(Q; x, y) = \sum_{\ell=0}^{\infty} Z^{(\ell)}(Q; t, q) Z^{(n+\ell)}(Q; t, q), \quad n \neq 0, \quad (3.18)$$

for $Z^{(0)} = 1$. Then, this formula enables us to derive the nontrivial invariance of superconformal indexes under $u \rightarrow u^{-1}$, and so we can rewrite the instanton contribution as a function of the $E_1 = SU(2)$ characters

$$\chi_{\mathbf{2s+1}}^{E_1}(u) = u^s + u^{s-1} + \dots + u^{-s+1} + u^{-s}. \quad (3.19)$$

Since the remaining factors inside the superconformal index are independent of u , this formula gives a non-trivial evidence of the conjecture that the index is a function of $SU(2)$ characters. Since the index is not a generic function of u but a function of $u^s + u^{-s}$, it is suggested that the flavor symmetry is enhanced from $U(1)$ to $SU(2)$ in all instanton numbers.

Using the defining equation (3.9), we can easily show

$$Z_{\text{pert}}(Q_F^{-1}, t^{-1}, q^{-1}) = Z_{\text{pert}}(Q_F^{-1}, t, q), \quad (3.20)$$

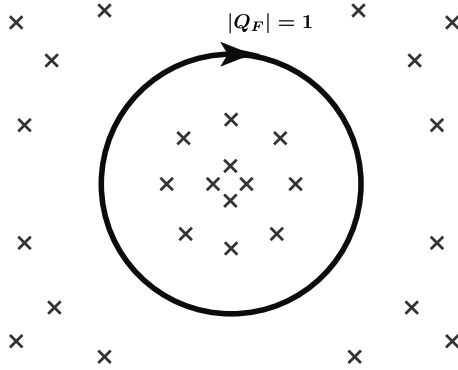


Figure 2: In the limit $|x| \ll 1$ the pole region of the integrand of the formula (3.21) splits into near the origin and faraway region.

and then we get the following expression

$$I_{5d}^{N_f=0}(u; x, y) = \mathcal{M}(x, y)^2 \sum_{n=0}^{\infty} (u^n + u^{-n}) \times \oint_{|Q|=1} \frac{dQ}{4\pi i Q} Z_{\text{pert}}(Q; x, y) Z_{\text{pert}}(Q^{-1}; x, y) F^{(n)}(Q; x, y). \quad (3.21)$$

To evaluate the index, we first expand this integrand into the positive-power series of x , and then we collect the Q -independent part of it. We can also compute the index by evaluating the contour integration directly. As we will see below, all the poles in the integrand of our formula (3.21) are located inside or outside of the unit circle $|Q| = 1$ in the parameter region $|x| \ll 1$ as **Fig.2**. We can therefore compute the residue integration by picking up all the poles inside the unit circle, and we can evaluate the integral without any regularization or ambiguity in $|x| \ll 1$.

Let us move on to explanation of how this formula works. In $|x| \ll 1$ we have the

unique power series expansion of the perturbative partition function in Q_F and x

$$\begin{aligned}
Z_{\text{pert}} &= \prod_{i,j} (1 - Q_F q^i t^{j-1})^{-1} (1 - Q_F q^{i-1} t^j)^{-1} \\
&= \prod_{i,j} (1 - Q_F q^i t^{-j}) (1 - Q_F q^{i-1} t^{1-j}) \\
&= 1 - Q_F \left(1 + \chi_2(y)x + (\chi_2(y))^2 x^2 + ((\chi_2(y))^3 - \chi_2(y))x^3 + \dots \right) \\
&\quad + Q_F^2 \left(\chi_2(y)x + (1 + (\chi_2(y))^2)x^2 + (2(\chi_2(y))^3 - \chi_2(y))x^3 + \dots \right) \\
&\quad - Q_F^3 (x^2 + (\chi_2(y))^3 x^3 + \dots) + \dots, \tag{3.22}
\end{aligned}$$

and therefore we obtain the power series expression of the perturbative contribution to the index $Z_{\text{pert}}(Q_F, t, q)Z_{\text{pert}}(Q_F^{-1}, t, q)$. The instanton contribution is more complicated, however, we can check that power expansion in $|x| \ll 1$ implies the unique Q_F -series expression. This is because a Nekrasov partition function is a summation over rational functions and each pole structure of a rational function takes the form $(1 - Q_F^l x^m y^n)^{-1}$. We therefore expand these factors into the following geometric power series

$$\frac{1}{1 - Q_F^l x^m y^n} = 1 + Q_F^l x^m y^n + Q_F^{2l} x^{2m} y^{2n} + \dots, \quad \text{if } m > 0, \tag{3.23}$$

$$\frac{1}{1 - Q_F^l x^m y^n} = -Q_F^{-l} x^{-m} y^{-n} - Q_F^{-2l} x^{-2m} y^{-2n} + \dots \quad \text{if } m < 0, \tag{3.24}$$

and then we obtain a positive power expansion in x . However, if $m = 0$ factor appeared, this algorithm would not work because this criterion for getting x series does not implies any preferred series expansion of $(1 - Q_F^l y^n)^{-1}$. Fortunately, some cancellation mechanism eliminates such seeming poles. For instance, the one-instanton part of $SU(2)$ partition function is

$$\frac{q}{t} u \frac{1}{(1-q)(1-t^{-1})} \left(\frac{1}{(1 - Q_F^{-1})(1 - Q_F t^{-1} q)} + \frac{1}{(1 - Q_F)(1 - Q_F^{-1} t^{-1} q)} \right), \tag{3.25}$$

and these two fixed point contributions have a common problematic pole $(1 - Q_F)^{-1}$. But we can collect these two contributions into the following single rational functions

$$\frac{u Q_F (q + t)}{(1 - q)(1 - t)(Q_F - t/q)(Q_F - q/t)}. \tag{3.26}$$

Therefore one-instanton partition function does not contain any problematic pole and we can obtain the unique series expansion. This idea works also for higher instanton numbers, and the denominator of the two-instanton partition function for instance is proportional to

$$(Q_F - t/q)(Q_F - t/q^2)(Q_F - t^2/q)(Q_F - q/t)(Q_F - q/t^2)(Q_F - q^2/t), \quad (3.27)$$

and there is no pole at $Q_F = (qt)^n$. This fact also holds for the Yang-Mills theory with the Chern-Simons level $m = 1, 2$. We can thus determine the expansion of the instanton contribution in the variables x and Q_F . This series expression immediately implies the superconformal index through our formula.

Pole structure and AGT relation

As we have observed experimentally, there is no problematic pole $(1 - Q_F(tq)^n)$ in the denominator of the instanton partition functions. We can verify this fact rigorously using the AGT relation [25, 26]. The AGT relation claims that the Nekrasov partition function of a 4d $\mathcal{N} = 2$ gauge theory is equal to the conformal block of the corresponding 2d CFT. The instanton number is the conformal level in the 2d theory. The denominator in the k -instanton partition function, which is the level- k part of the conformal block, is the level- k Kac determinant. This determinant

$$\det K_k = c_k \prod_{1 \leq r, s \leq k} (h - h_{r,s})^{P(k-rs)} \quad (3.28)$$

where P is the number $\prod_{n=1}^k (1 - q^n)^{-1} = \sum_{m=0}^k P(m)q^m$, h is the conformal dimension, and the zeros of the determinant is

$$h_{r,s} = \frac{Q^2 - (rb + s/b)^2}{4}, \quad Q = b + \frac{1}{b}. \quad (3.29)$$

We parametrize the central charge as $c = 1 + 6Q^2$, and the Ω -background is related to the Liouville background charge $b = \epsilon_1$ and $1/b = \epsilon_2$.

Let us apply this fact to our problem. If the problematic pole $(1 - Q_F(tq)^n)$ exists in the five-dimensional partition function, it implies the following pole in the denominator of the partition function in the four-dimensional limit:

$$(2a - n(b - 1/b))(2a + n(b - 1/b)). \quad (3.30)$$

Notice the symmetry $n \rightarrow -n$ in the denominator. The AGT relation is satisfied under the identification between the conformal dimension and the Coulomb branch parameter $h = Q^2/4 - a^2$, this problematic factor corresponds to

$$h - \frac{Q^2 - (nb - n/b)^2}{4}. \quad (3.31)$$

Such a pole at $h_{n,-n}$ is obviously absent from the Kac determinant, and therefore we conclude that the instanton partition function is free from any problematic poles.

Actually, our logic works only if the five-dimensional uplift of the four-dimensional denominator is just the q -number extension $\det(K_k) \propto \prod (2a - a_{rs})^{p_{rs}} \rightarrow \prod (1 - e^{R(2a - a_{rs})})^{p_{rs}}$ of the Kac determinant. This fact is verified for pure $SU(2)$ Yang-Mills theory with zero Chern-Simons level $m_{\text{eff}} = 0$. The Kac determinant of the q -Virasoro algebra [33, 34] is given by

$$\det K_k = \prod_{1 \leq rs \leq k} \left(\frac{(1 - q^r)(1 - t^r)}{q^r + r^r} \right)^{P(k-rs)} (\lambda^2 - \lambda_{r,s}^2)^{P(k-rs)}, \quad (3.32)$$

where $\lambda_{r,s} = q^{-s/2}t^{r/2} + q^{s/2}t^{-r/2}$. The AGT relation in five dimensions [35] suggests the following map between the Coulomb branch parameter and the dimension of the corresponding primary λ

$$\lambda = Q_F^{\frac{1}{2}} + Q_F^{-\frac{1}{2}}. \quad (3.33)$$

The Kac determinant therefore takes the form

$$(\lambda^2 - \lambda_{r,s}^2)^{P(k-rs)} = (Q_F - q^s t^{-r})^{P(k-rs)} (1 - q^{-s} t^r Q_F^{-1})^{P(k-rs)}, \quad (3.34)$$

and no factor in question appears. There is therefore no problematic pole in the denominators of Nekrasov partition functions if the AGT relation is satisfied. The idea and formulation toward the proof of the five-dimensional AGT relation in the case of $N_c = 2$, $N_f = 0$ and $m = 0$ is proposed in [36, 37].

\tilde{E}_1 theory and local \mathbb{F}_1 partition function

Let us move on to the computation of the index for the UV fixed point theory \tilde{E}_1 which arises from the $SU(2)$ Yang-Mills theory with non-vanishing theta term $\vartheta \equiv 1 \pmod{2}$. It is straightforward to compute the superconformal index in this case.

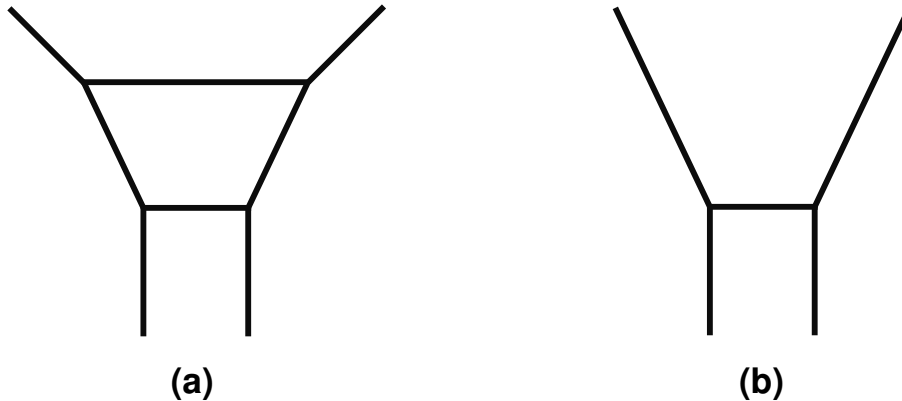


Figure 3: The local \mathbb{F}_2 geometry and its six-dimensional contribution. (a) 5-brane web for the local \mathbb{F}_2 geometry. (b) on the coincided two 5-branes, additional six-dimensional degrees of freedom appear.

The full Nekrasovpartition function is

$$Z_{\mathbb{F}_1}(Q_B, Q_F; t, q) = \mathcal{M}(x, y) Z_{\text{pert}}^{N_f=0}(Q_F; t, q) Z_{m=1}^{N_f=0}(Q_B, Q_F; t, q), \quad (3.35)$$

where the instanton partition function is

$$Z_{m=1}^{N_f=0}(Q_B, Q_F; t, q) = \sum_{\vec{Y}} \left(-u \frac{q}{t}\right)^{|\vec{Y}|} \frac{Q_F^{\frac{|Y_1|-|Y_2|}{2}} \prod_{\alpha} t^{-\frac{\|Y_{\alpha}\|^2}{2}} q^{\frac{\|Y_{\alpha}\|^2}{2}}}{\prod_{\alpha, \beta=1}^2 N_{Y_{\alpha}, Y_{\beta}}(Q_{\beta\alpha}; t, q)} = \sum_{n=0}^{\infty} u^n Z_{m=1}^{(n)}(Q_F, t, q). \quad (3.36)$$

In this case

$$Z_{m=1}^{(n)}(Q_F; t, q) = \sum_{|\vec{Y}|=n} \frac{\left(-\frac{q}{t}\right)^{|\vec{Y}|} Q_F^{\frac{|Y_1|-|Y_2|}{2}} \prod_{\alpha=1,2} t^{-\frac{\|Y_{\alpha}\|^2}{2}} q^{\frac{\|Y_{\alpha}\|^2}{2}}}{\prod_{\alpha, \beta=1}^2 N_{Y_{\alpha}, Y_{\beta}}(Q_{\beta\alpha}; t, q)}. \quad (3.37)$$

By applying this instanton partition function we can compute the superconformal index as the formula in the previous example.

3.2 New 5d SCFT from local \mathbb{F}_2 ?

The Yang-Mills theories with effective Chern-Simons levels $m = 0, 1$ are engineered geometrically by the local Calabi-Yau threefold over the Hirzebruch surfaces \mathbb{F}_m . Actually these two theories corresponding to the two allowed values of the discrete theta angle

$\vartheta = 0, \pi$. The UV fixed point theories of them are the known superconformal field theories: E_1 and \tilde{E}_1 , We have studied the superconformal indexes of them.

There exists an another geometry in the sequence of the Hirzebruch surfaces, known as \mathbb{F}_2 . The local geometry over this surface gives the Yang-Mills theory with $m = 2$, however no UV fixed point theory is known so far in the strong coupling limit. One possibility is hence that this Yang-Mills theory is meaningless in the strongly-coupled UV region, and it is only a cut-off theory and needs the UV completion by adding additional degrees of freedom to make sense. Actually the brane construction in the strong coupling limit provides additional massless states since the two parallel 5-branes coincides and the six-dimensional massless excitations on them would couple to the boundary five-dimensional theory **Fig.3**. Another possibility is that the six-dimensional degrees of freedom are actually decoupled from the boundary five-dimensional theory, and we get a new five-dimensional fixed point theory at the strong coupling limit. We can not make a judgement only from the brane realization. In the following we find the solution to the question by computing the superconformal index.

The Nekrasov partition function for $SU(2)$ pure Yang-Mills theory with the Chern-Simons level 2, which corresponds to the topological string partition function for the local \mathbb{F}_2 geometry, takes the form

$$Z_{\mathbb{F}_2}(Q_B, Q_F; t, q) = \mathcal{M}(x, y) Z_{\text{pert}}^{N_f=0}(Q_F; t, q) Z_{m=2}^{N_f=0}(Q_B, Q_F; t, q) \quad (3.38)$$

$$Z_{m=2}^{N_f=0}(Q_B, Q_F; t, q) = \sum_{\vec{Y}} \left(u \frac{q}{t}\right)^{|\vec{Y}|} \frac{Q_F^{|Y_1| - |Y_2|} \prod_{\alpha} t^{-\|Y_{\alpha}^t\|^2} q^{\|Y_{\alpha}\|^2}}{\prod_{\alpha, \beta=1}^2 N_{Y_{\alpha}, Y_{\beta}}(Q_{\beta\alpha}; t, q)}. \quad (3.39)$$

In order to use this formula for our index computation, we introduce the relation between the physical parameters

$$Q_F = e^{2ia}, \quad Q_B = u. \quad (3.40)$$

This Nekrasov partition function then implies the following superconformal index through

the formula (2.8)

$$\begin{aligned}
I_{\mathbb{F}_2}(u; x, y) = & 1 + x^2 + 2\chi_2(y)x^3 + (2\chi_3(y) + 1)x^4 \\
& + \left\{ -\left(u + \frac{1}{u}\right)(\chi_3(y) - 1) - 1 + \chi_3(y) + 2\chi_4(y) \right\} x^5 \\
& + \left\{ -\left(u + \frac{1}{u}\right)(\chi_3(y) + 2) - 1 + 3\chi_3(y) + 2\chi_5(y) \right\} x^6 \\
& + \left\{ -\left(u + \frac{1}{u}\right)(2\chi_2(y) + \chi_4(y)) + \chi_2(y) + 4\chi_4(y) + 2\chi_6(y) \right\} x^7 + \\
& + \left\{ 6 + 6\chi_3(y) + 7\chi_5(y) + 2\chi_7(y) \right. \\
& \left. - \left(u + \frac{1}{u}\right)(-1 + 3\chi_3(y) + \chi_5(y)) + \left(u^2 + \frac{1}{u^2}\right) \right\} x^8 + \dots \quad (3.41)
\end{aligned}$$

Many coefficients of the terms in this expanded index are negative. In general, the negative terms of an index correspond to the fermionic contributions. But the above expression seems to be unusual because all the superconformal indexes computed in [5] have only a few negative terms. In the above case, we can save this unnaturalness technically just by redefining the fugacity u as

$$u \rightarrow -u. \quad (3.42)$$

We however find a more natural solution to this problem. Our claim is that the above expression contains certain extra contribution in addition to that from 5d fixed point theory. In the following we divide an unnecessary contribution from the index, and we then obtain the ‘‘proper’’ five-dimensional superconformal index whose almost all the coefficients are positive.

Then, what is this extra contribution to the superconformal index? In the 5-brane construction **Fig.3**, there exist two parallel 5-branes which coincide with each other at the superconformal fixed point. This fact suggests that the theory coming from a 5-brane web can contain some massless degrees of freedom on the stack of the coincided 5-branes. This has been pointed out in [27, 28] by studying the five-dimensional uplift of the Gaiotto T_N theory. We can also identify the extra contribution which makes original index non-proper as the six-dimensional degrees of freedom on the 5-brane stack. In the language of the Calabi-Yau compactification, this stack of parallel 5-branes is dual to the local structure $\mathcal{O}(0) \oplus \mathcal{O}(-2) \rightarrow \mathbb{CP}^1$ in the local \mathbb{F}_2 geometry **Fig.3(b)**. The moduli

space of the BPS states of the compactification onto this geometry, that is the moduli space of M2-branes wrapping the two cycles, is then non-compact because of the flat $\mathcal{O}(0)$ direction, and the resulting BPS states do not form full spin content⁵. To eliminate the extra contribution caused by this non-compactness, we factor the index (or partition function) of the local $\mathcal{O}(0) \oplus \mathcal{O}(-2) \rightarrow \mathbb{CP}^1$ geometry out of the index (or partition function). This procedure is a simplified version of that for $N_f \geq 2$ theories found in [27, 28], and this extra factor coming from the non-full spin content is a five-dimensional analogue of the proportional coefficient of the AGT relation between Liouville correlators and the partition functions of gauge theories in four dimensions.

Let us move on to the computation of this extra prefactor. Since this non-full spin content contribution is the partition function on the local geometry **Fig.3(b)**, the refined topological vertex formalism provides the explicit expression. We can therefore get the closed expression for the partition function of this non-full spin content by using the refined topological vertex formalism

$$\begin{aligned} Z_{\mathcal{O}(0,-2)_{\mathbb{CP}^1}} &= Z_{\text{non-full}}(u; x, y) \\ &= \sum_Y C_{\emptyset\emptyset Y}(q, t)(-Q_B)^{|Y|} f_Y(q, t) C_{\emptyset\emptyset Y^t}(t, q) \\ &= \prod_{i,j=1}^{\infty} \frac{1}{(1 - ut^{i-1}q^j)}. \end{aligned} \quad (3.43)$$

In order to get the last equality we apply the formula (4.14). We employ the following connected expression to get positive power expansion in x

$$Z_{\text{non-full}}(u; x, y) = \prod_{i,j=1}^{\infty} (1 - ut^{-i}q^j) = \prod_{i,j=1}^{\infty} (1 - ux^{i+j}y^{-i+j}). \quad (3.44)$$

The proper instanton contribution to the 5d theory is then given by the following normalized partition function

$$\begin{aligned} Z_{m=2}^{\text{full spin}}(u, Q; x, y) &= \frac{Z_{m=2}^{N_f=0}(u, Q; x, y)}{Z_{\text{non-full}}(u; x, y)} = 1 + ux^2 + u\chi_2(y)x^3 + \left(u \left(Q + 1 + \frac{1}{Q} + \chi_3(y) \right) \right. \\ &\quad \left. + u^2 \right) x^4 + \left(u\chi_2(y) \left(Q + 1 + \frac{1}{Q} \right) + u\chi_4(y) + u^2\chi_2(y) \right) x^5 + \dots \end{aligned} \quad (3.45)$$

⁵The author thanks his collaborators on the recent paper [27]: L.Bao, V.Mitev, E.Pomoni and F.Yagi. He also thanks C.Kozçaz for discussion on this problem through the collaborators. See also the references [11, 6].

Then the normalized superconformal index is

$$\begin{aligned} \tilde{I}_{\mathbb{F}_2}(u; x, y) &= \mathcal{M}(x, y) \oint \frac{dQ_F}{4\pi i Q_F} |Z_{\text{pert}}^{N_f=0}(Q_F; x, y)|^2 \\ &\quad \times Z_{m=2}^{\text{full spin}}(u, Q_F; x, y) Z_{m=2}^{\text{full spin}}(u^{-1}, Q_F^{-1}; x, y). \end{aligned} \quad (3.46)$$

Notice that the perturbative contribution is not affected by the ‘‘Chern-Simons level’’ $m = 2$. By computing the residue integral, we find

$$\begin{aligned} \tilde{I}_{\mathbb{F}_2}(u; x, y) &= 1 + \chi_{\mathbf{3}}^{E_1}(u)x^2 + \chi_2(y) (1 + \chi_{\mathbf{3}}^{E_1}(u)) x^3 + \left\{ \chi_3(y) (1 + \chi_{\mathbf{3}}^{E_1}(u)) + 1 + \chi_{\mathbf{5}}^{E_1}(u) \right\} x^4 \\ &+ \left\{ \chi_4(y) (1 + \chi_{\mathbf{3}}^{E_1}(u)) + \chi_2(y) (1 + \chi_{\mathbf{3}}^{E_1}(u) + \chi_{\mathbf{5}}^{E_1}(u)) \right\} x^5 \\ &+ \left\{ \chi_5(y) (1 + \chi_{\mathbf{3}}^{E_1}(u)) + 2\chi_3(y) (1 + \chi_{\mathbf{3}}^{E_1}(u) + \chi_{\mathbf{5}}^{E_1}(u)) - 1 + \chi_{\mathbf{3}}^{E_1}(u) + \chi_{\mathbf{7}}^{E_1}(u) \right\} x^6 \\ &+ \left\{ \chi_6(y) (1 + \chi_{\mathbf{3}}^{E_1}(u)) + \chi_4(y) (2 + 4\chi_{\mathbf{3}}^{E_1}(u) + 2\chi_{\mathbf{5}}^{E_1}(u)) \right. \\ &\quad \left. + \chi_3(y) (1 + 3\chi_{\mathbf{3}}^{E_1}(u) + 2\chi_{\mathbf{5}}^{E_1}(u) + \chi_{\mathbf{7}}^{E_1}(u)) \right\} x^7 + \left\{ \chi_7(y) (1 + \chi_{\mathbf{3}}^{E_1}(u)) \right. \\ &\quad \left. + \chi_5(y) (4 + 5\chi_{\mathbf{3}}^{E_1}(u) + 3\chi_{\mathbf{5}}^{E_1}(u)) + \chi_3(y) (2 + 7\chi_{\mathbf{3}}^{E_1}(u) + 3\chi_{\mathbf{5}}^{E_1}(u) + 2\chi_{\mathbf{7}}^{E_1}(u)) \right. \\ &\quad \left. + 3 + 2\chi_{\mathbf{3}}^{E_1}(u) + 2\chi_{\mathbf{5}}^{E_1}(u) + \chi_{\mathbf{9}}^{E_1}(u) \right\} x^8 + \left\{ \chi_8(y) (1 + \chi_{\mathbf{3}}^{E_1}(u)) \right. \\ &\quad \left. + \chi_6(y) (4 + 7\chi_{\mathbf{3}}^{E_1}(u) + 3\chi_{\mathbf{5}}^{E_1}(u)) + \chi_4(y) (6 + 10\chi_{\mathbf{3}}^{E_1}(u) + 6\chi_{\mathbf{5}}^{E_1}(u) + 3\chi_{\mathbf{7}}^{E_1}(u)) \right. \\ &\quad \left. + \chi_2(y) (4 + 7\chi_{\mathbf{3}}^{E_1}(u) + 4\chi_{\mathbf{5}}^{E_1}(u) + 2\chi_{\mathbf{7}}^{E_1}(u) + \chi_{\mathbf{9}}^{E_1}(u)) \right\} x^9 + \dots \end{aligned} \quad (3.47)$$

This result is precisely equal to that of E_1 SCFT $I_{\mathbb{F}_0}(u; x, y)$ computed in [5, 6], and so we can conclude that the 5d fixed point theory associated with local \mathbb{F}_2 is precisely the well known E_1 SCFT arising from $SU(2)$ pure Yang-Mills theory without discrete theta term $m = 0^6$. This result is natural from the perspective of Calabi-Yau compactification. The surface \mathbb{F}_2 is a complex structure deformation of the surface \mathbb{F}_0 , and the topological string partition function, namely the Nekrasov partition function, is independent of such a deformation. The partition function of \mathbb{F}_2 is then equal to that of \mathbb{F}_0 if this independence on the complex structure is satisfied. The refined topological string partition function

⁶This fact has been conjectured in [38] very recently based on the one-instanton partition function, and our result provides a multi-instanton justification of this conjecture.

however can jump under the complex structure deformation, and the surfaces $\mathbb{F}_{0,2}$ are different in the Nekrasov partition function. This difference is, however, very simple as we found

$$Z_{\text{c.s.d.of } \mathbb{F}_0} = Z_{\text{non-full}} \cdot Z_{\mathbb{F}_0}, \quad (3.48)$$

where $Z_{\text{c.s.d.of } \mathbb{F}_0}$ is the partition function of the complex structure deformation of \mathbb{F}_0 , namely \mathbb{F}_2 . The jump of the refined Gopakumar-Vafa invariants therefore can be collected into the overall factor $Z_{\text{non-full}}$ which is the non-full spin content of the \mathbb{F}_2 theory. This structure is very similar to that of the wall crossing phenomena, and it would be very interesting if we can justify our expectation (3.48) based on the theory of the refined Gopakumar-Vafa invariants and its wall crossing.

Moreover, our result gives an evidence for the conjecture that the contribution from non-compact parallel 5-branes can be eliminated by factoring the partition function of the non-full spin content out. This conjecture is originally based on the comparison between $SU(2)$ $2 \leq N_f \leq 5$ gauge theory and the expected E_{N_f+1} UV fixed point. Our argument in this article is based on the relation between the discrete theta terms and the two known fixed point theories E_1 and \tilde{E}_1 . It therefore provides an another evidence of the above mentioned conjecture on the contribution of the non-full spin content.

Our result suggests that the correspondence between generic Hirzebruch surface and UV fixed point theories. For $m \geq 3$, the Hirzebruch surface \mathbb{F}_m is not Fano, and is specified by a non-convex fan. We can expect that the local geometry over \mathbb{F}_m is related to the E_s theory as $m \equiv s \pmod{2}$. By computing the partition function for \mathbb{F}_m and its non-full spin content along the line of [27, 28], we can expect

$$I_{\mathbb{F}_s}(u; x, y) = \frac{I_{\mathbb{F}_{2N+s}}(u; x, y)}{I_{\mathbb{F}_{2N+s}}^{\text{non-full}}(u; x, y)}, \quad s = 0, 1, \quad N = 0, 1, 2, 3, \dots \quad (3.49)$$

We leave the detailed check of this relation to future study.

3.3 E_2 theory

The local second del Pezzo surface dP_2 engineers $SU(2)$ gauge theory with single fundamental flavor, and it was shown that the topological string partition function for the geometry is exactly equal to the five-dimensional Nekrasov partition function for the

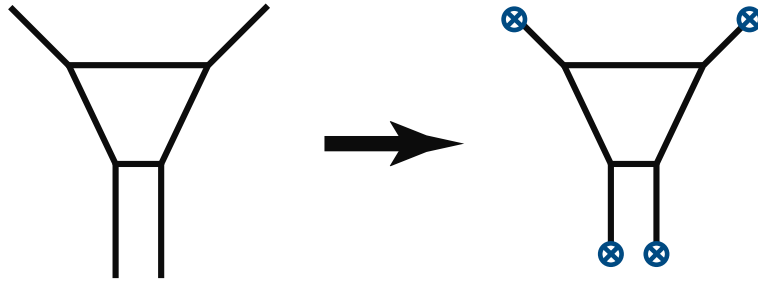


Figure 4: We can regularize the 5-brane web by terminating the external branes on 7-branes. The 7-branes are illustrated by the blue circles. We conjecture that the regularized theory can be computed by eliminating the contribution from the non-full-spin content, and the resulting 5d theory is equal to the local \mathbb{F}_0 theory, i.e. the E_1 SCFT.

gauge theory. The full Nekrasov partition function is

$$Z_{\text{pert}}^{N_f=1}(Q_F, z; t, q) = \mathcal{M}(x, y) \prod_{i,j=1}^{\infty} (1 - Q_F q^i t^{j-1})^{-1} (1 - Q_F q^{i-1} t^j)^{-1} \\ \times (1 - Q_E t^{i-\frac{1}{2}} q^{j-\frac{1}{2}}) (1 - Q_F Q_E t^{i-\frac{1}{2}} q^{j-\frac{1}{2}} t) \quad (3.50)$$

$$Z_{\text{inst}}^{N_f=1}(uQ_F, Q_F, z; t, q) = \sum_{\vec{Y}} \left(u \frac{q}{t}\right)^{|\vec{Y}|} \frac{Z_{\text{fund}}^{\vec{Y}}(Q_a, Q_m = z; t, q)}{Z_{\text{vec}}^{\vec{Y}}(Q_a; t, q)}, \quad (3.51)$$

where $Q_E = Q_m / \sqrt{Q_F} = z / \sqrt{Q_F}$. The vector multiplet contribution takes the form

$$Z_{\text{vec}}^{\vec{Y}}(Q_a; t, q) = \frac{\left(\frac{q}{t}\right)^{\frac{N_c |\vec{Y}|}{2}}}{\prod_{\alpha, \beta} N_{\alpha, \beta}^{\vec{Y}}(Q_{\beta \alpha}; t, q)}. \quad (3.52)$$

We can therefore show the following symmetry of this factor

$$Z_{\text{vec}}^{(Y_1, Y_2)}(Q_a^{-1}; t, q) = Z_{\text{vec}}^{(Y_2, Y_1)}(Q_a; t, q), \quad Z_{\text{fund}}^{(Y_1, Y_2)}(Q_a^{-1}, Q_m; t, q) = Z_{\text{fund}}^{(Y_2, Y_1)}(Q_a, Q_m; t, q). \quad (3.53)$$

This property leads to the relation $Z_{\text{inst}}^{N_f=1}(uQ_F^{-1}, Q_F^{-1}, z; t, q) = Z_{\text{inst}}^{N_f=1}(uQ_F, Q_F, z; t, q)$, and it implies the enhancement $U(1) \rightarrow SU(2)$ of the flavor symmetry associated with the fugacity u as the case of pure Yang-Mills theory.

We can show an expression of the superconformal index very similar to (3.21). The only difference is that the partition function in the expression is not that of pure Yang-Mills theory but that of $N_f = 1$ theory in this case. The parametrization is $Q_B = ue^{2ia}$, $Q_F = e^{2ia}$ and $Q_E = ze^{-ia}$ [6], and so the integration is taken over Q_F .

3.4 E_3 theory

Let us consider $SU(2)$ gauge theory with two fundamental hypermultiplets. The Nekrasov partition function is

$$Z_{\text{inst}}^{N_f=2}(uQ_F, Q_F, z_1, z_2; t, q) = \sum_{\vec{Y}} \left(u \frac{q}{t}\right)^{|\vec{Y}|} \frac{\prod_{f=1,2} Z_{\text{fund}}^{\vec{Y}}(Q_a, Q_{mf}; t, q)}{\prod_{\alpha,\beta=1}^2 N_{Y_\alpha, Y_\beta}(Q_{\beta\alpha}; t, q)}, \quad (3.54)$$

where we use the following parametrization

$$Q_{m1} = z_1 z_2, \quad Q_{m2} = \frac{z_2}{z_1}. \quad (3.55)$$

It is easy to show the following invariance which suggests the enhancement $U(1) \rightarrow SU(2)$

$$Z_{\text{inst}}^{N_f=2}(uQ_F^{-1}, Q_F^{-1}, z_1, z_2; t, q) = Z_{\text{inst}}^{N_f=2}(uQ_F, Q_F, z_1, z_2; t, q). \quad (3.56)$$

Since the partition function is invariant under the exchange of the masses $Q_{m1} \leftrightarrow Q_{m2}$, we find the invariance

$$Z_{\text{inst}}^{N_f=2}(uQ_F, Q_F, z_1^{-1}, z_2; t, q) = Z_{\text{inst}}^{N_f=2}(uQ_F, Q_F, z_1, z_2; t, q). \quad (3.57)$$

This suggests that the abelian flavor symmetry associated with $z_1 = \sqrt{Q_{m1}/Q_{m2}}$ is also enhanced to $SU(2)$. The following enhancement is therefore manifest in our formula,

$$U(1)_u \times U(1)_{z_1} \times U(1)_{z_2} \rightarrow SU(2) \times SU(2) \times U(1) \subset SU(2) \times SU(3) = E_3. \quad (3.58)$$

To get the full enhancement, we have to eliminate the extra contribution as the case of \mathbb{F}_2 geometry. See [27, 28] for details on this issue.

4 Discussion

In this article we have discussed on the enhanced flavor symmetry of five-dimensional UV fixed point theories through the superconformal index. We gave the expression of superconformal indexes in which the dependence on the special combination of instanton fugacity $u^s + u^{-s}$ becomes manifest, and then we can find that the index is a function of the $SU(2)$ characters $u^s + u^{s-1} + \dots + u^{-s}$. We extended our argument for some cases of higher-rank flavor symmetry by using the structure of the Nekrasov partition function of SQCD.

We also compute the superconformal index associated with the local \mathbb{F}_2 geometry. It had been known that the local Hirzebruch surfaces \mathbb{F}_0 and \mathbb{F}_1 correspond to the two known superconformal field theories with rank one flavor symmetry E_1 and \tilde{E}_1 . There exist, however, another Calabi-Yau manifolds and the corresponding 5-brane web configurations which lead to rank one flavor symmetry and one-dimensional Coulomb branch. The local second Hirzebruch surface \mathbb{F}_2 is a typical example, and it had been an open problem what is the superconformal field theory which appears at the UV fixed point of the \mathbb{F}_2 theory. In this paper we pointed out that the five-dimensional theory arising from M-theory on the local \mathbb{F}_2 geometry contains some extra degrees of freedom in the nature of an six-dimensional excitation. We proposed the method to remove the extra contribution from the Nekrasov partition function, and then found that the resulting superconformal index is identical with that of the \mathbb{F}_0 theory. Therefore the superconformal field theory for \mathbb{F}_2 geometry should be precisely the E_1 theory which is the UV fixed point theory of $SU(2)$ pure Yang-Mills theory with vanishing discrete theta term. This fact was pointed out very recently in [38] based on the one instanton partition function, and our argument is based on the multi-instanton computation and the superconformal index.

It would be interesting to extend our idea precisely to E_{N_f+1} theories for $N_f = 2, \dots, 7$. Another interesting application of our result would be the research on the unknown relation between UV fixed point theories associated with various 5-brane web configurations. Since we can extract the proper superconformal index from the topological vertex partition function by removing the extra contribution which does not form full spin content, we can check the relation between fixed point theories by comparing quantitatively the superconformal indexes. We leave these problems for future study.

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Appendix A : Building blocks of the Nekrasov partition function

A-1 Young diagram

The arm and leg length are defined

$$a_Y(i, j) = Y_i - j, \quad \ell_Y(i, j) = Y_j^t - i, \quad a'(i, j) = j - 1, \quad \ell'(ij) = i - 1. \quad (4.1)$$

We introduce the norms of a Young diagram as $|Y| := \sum_{i=1}^{d(Y)} Y_i$ and $\|Y\|^2 := \sum_{i=1}^{d(Y)} Y_i^2$.

$$\begin{aligned} n(Y) &= \sum_{s \in Y} \ell_Y(s) = \sum_{s \in Y} \ell'(s) = \frac{\|Y^t\|^2 - |Y|}{2} \\ \kappa_Y &= 2(n(Y^t) - n(Y)). \end{aligned} \quad (4.2)$$

A-2 : symmetric functions

The refined MacMahon function is

$$M(t, q) = \prod_{i, j=1}^{\infty} (1 - q^i t^{j-1})^{-1}. \quad (4.3)$$

The following specialized Macdonald function play a role in the refined topological vertex formalism:

$$\tilde{Z}_R(t, q) = \prod_{i=1}^{d(R)} \prod_{j=1}^{R_i} (1 - q^{R_i - j} t^{R_j^t - i + 1})^{-1}. \quad (4.4)$$

The Macdonald functions are basis of the ring of the symmetric functions over the field of rational functions $\mathbb{Q}(q, t)$. We can define the Macdonald functions uniquely as the functions

$$P_\lambda(x; q, t) = m_\lambda(x) + \sum_{\mu \prec \lambda} u_{\lambda\mu} m_\mu(x), \quad u_{\lambda\mu} \in \mathbb{Q}(t, q), \quad (4.5)$$

which satisfy the orthogonality

$$\langle P_\mu, P_\nu \rangle_{q, t} = 0 \quad \text{if } \mu \neq \nu. \quad (4.6)$$

The inner product is defined by using that for the power-sum symmetric functions

$$\langle p_\mu, p_\nu \rangle_{q,t} = \delta_{\mu\nu} z_\lambda \prod_{i=1}^{\lambda_i} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}. \quad (4.7)$$

The norm of the Macdonald function is

$$\langle P_\lambda, P_\lambda \rangle_{q,t} = \prod_{s \in \lambda} \frac{1 - q^{a(s)+1} t^{\ell(s)}}{1 - q^{a(s)} t^{\ell(s)+1}} \equiv \left(\frac{q}{t}\right)^{\frac{|\lambda|}{2}} \frac{1}{b_\lambda(q,t)}. \quad (4.8)$$

The Macdonald function satisfies the following Cauchy formulas

$$\sum_{\lambda} b_\lambda(q,t) P_\lambda(x; q, t) P_\lambda(y; q, t) = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{q}{t}\right)^{\frac{n}{2}} \frac{1 - t^n}{1 - q^n} p_n(x) p_n(y) \right), \quad (4.9)$$

$$\sum_{\lambda} P_\lambda(x; q, t) P_{\lambda^t}(y; t, q) = \prod_{i,j} (1 + x_i y_j). \quad (4.10)$$

These relations play the key role to take a summation over Young diagrams and get a closed expression.

The principal specialization of the Macdonald function is important to study the Nekrasov formulas

$$P_\lambda(t^\rho; q, t) = \prod_{s \in \lambda} \frac{-t^{\frac{1}{2}} q^{a(s)}}{1 - q^{a(s)} t^{\ell(s)+1}}, \quad (4.11)$$

$$P_{\lambda^t}(q^\rho; t, q) = \prod_{s \in \lambda} \frac{q^{-a(s) - \frac{1}{2}}}{1 - q^{-a(s) - 1} t^{-\ell(s)}}. \quad (4.12)$$

This specialization is related to \tilde{Z}_λ as

$$\tilde{Z}_\lambda(t, q) = \left(-\sqrt{\frac{q}{t}}\right)^{|\lambda|} q^{-\frac{\|\lambda\|^2}{2}} P_\lambda(t^\rho; q, t). \quad (4.13)$$

By using the Cauchy formula (4.9) for $x = y = t^\rho$, we can obtain

$$\sum_{\lambda} \left(Q \sqrt{\frac{t}{q}}\right)^{|\lambda|} q^{\|\lambda\|^2} \tilde{Z}_\lambda(t, q) \tilde{Z}_{\lambda^t}(q, t) = \prod_{i,j=1}^{\infty} \frac{1}{1 - Q t^{i-\frac{1}{2}} q^{j-\frac{1}{2}}}. \quad (4.14)$$

A-3 : Nekrasov partition function

Pure Yang-Mills theory

Let us introduce the following factor

$$\begin{aligned} N_{\alpha,\beta}^{\vec{Y}}(Q_{\beta\alpha}; t, q) &= \prod_{s \in Y_\alpha} \left(1 - Q_{\beta\alpha} t^{\ell_{Y_\beta}(s)} q^{a_{Y_\alpha}(s)+1}\right) \prod_{t \in Y_\beta} \left(1 - Q_{\beta\alpha} t^{-(\ell_{Y_\alpha}(t)+1)} q^{-a_{Y_\beta}(t)}\right) \\ &= \prod_{i,j=1}^{\infty} \frac{1 - Q_{\beta\alpha} t^{-Y_{\alpha,j}^t + i - 1} q^{-Y_{\beta,i} + j}}{1 - Q_{\beta\alpha} t^{i-1} q^j}. \end{aligned} \quad (4.15)$$

We also indicate the same function by the symbol $N_{Y_\alpha, Y_\beta}(Q_{\beta\alpha}; t, q)$. The vector multiplet contribution to the Nekrasov partition functions is,

$$Z_{\text{vec}}^{\vec{Y}} = \frac{\left(\frac{q}{t}\right)^{\frac{N_c |\vec{Y}|}{2}}}{\prod_{\alpha,\beta} N_{\alpha,\beta}^{\vec{Y}}(Q_{\beta\alpha}; t, q)}, \quad (4.16)$$

and the Chern-Simons term with the effective level m_{eff} contributes in the following form

$$Z_{\text{CS}}^{\vec{Y}, m_{\text{eff}}} = \left(\frac{q}{t}\right)^{\frac{m_{\text{eff}} |\vec{Y}|}{2}} \prod_{\alpha} Q_{\alpha}^{-m_{\text{eff}} |Y_{\alpha}|} \prod_{(i,j) \in Y_{\alpha}} t^{-m_{\text{eff}}(i-1)} q^{m_{\text{eff}}(j-1)}. \quad (4.17)$$

Then the five-dimensional Nekrasov partition function for $\mathcal{N} = 2$ $U(N_c)$ pure SYM theory with the Chern-Simons level $m = m_{\text{eff}}$ is

$$\begin{aligned} Z_{5\text{d}}^m(\vec{a}, \Lambda, R; \epsilon_1, \epsilon_2) &= \sum_{\vec{Y}} \mathfrak{Q}^{|\vec{Y}|} \left(\frac{q}{t}\right)^{\frac{(N_c+m)|\vec{Y}|}{2}} \frac{\prod_{\alpha} Q_{\alpha}^{-m|Y_{\alpha}|} \prod_{(i,j) \in Y_{\alpha}} t^{-m(i-1)} q^{m(j-1)}}{\prod_{\alpha,\beta} N_{\alpha,\beta}^{\vec{Y}}(Q_{\beta\alpha}; t, q)} \\ &= \sum_{\vec{Y}} \mathfrak{Q}^{|\vec{Y}|} \left(\frac{q}{t}\right)^{\frac{N_c |\vec{Y}|}{2}} \frac{\prod_{\alpha} Q_{\alpha}^{-m|Y_{\alpha}|} t^{-\frac{m \|Y_{\alpha}^t\|^2}{2}} q^{\frac{m \|Y_{\alpha}\|^2}{2}}}{\prod_{\alpha,\beta} N_{\alpha,\beta}^{\vec{Y}}(Q_{\beta\alpha}; t, q)}. \end{aligned} \quad (4.18)$$

We use the instanton factor $\mathfrak{Q} = (R\Lambda)^{2N_c}$ as the weight to count the instanton number, which is five-dimensional analogue of the dimensional transmutation.

Symmetry of the $SU(2)$ vector multiplet contribution

Let us consider the following n -instanton partition function of $SU(2)$ Yang-Mills theory

$$Z^{(n)}(Q, t, q) = \sum_{|\vec{Y}|=n} Z_{\text{vec}}^{\vec{Y}}(Q_{21} = Q). \quad (4.19)$$

This partition function enjoys the following symmetries

$$Z^{(n)}(Q^{-1}, t, q) = Z^{(n)}(Q, t, q), \quad Z^{(n)}(Q, t^{-1}, q^{-1}) = Z^{(n)}(Q, t, q). \quad (4.20)$$

The first relation follows from the following relation

$$\begin{aligned} Z_{\text{vec}}^{(Y_1, Y_2)}(Q_{21} = Q^{-1}) &= \left(\frac{q}{t}\right)^{|\vec{Y}|} \frac{1}{N_{Y_1 Y_1}(1) N_{Y_1 Y_2}(Q) N_{Y_2 Y_1}(Q^{-1}) N_{Y_2 Y_2}(1)} \\ &= Z_{\text{vec}}^{(Y_2, Y_1)}(Q_{21} = Q). \end{aligned} \quad (4.21)$$

The identity given in [21]

$$\sum_{s \in Y_\alpha} \ell_{Y_\beta}(s) = \sum_{j=1}^{Y_{\alpha,1}^t} \sum_{i=1}^{Y_{\alpha,j}^t} Y_{\beta,j}^t - \sum_{s \in Y_\alpha} i = \sum_{j=1}^{\infty} Y_{\alpha,j}^t Y_{\beta,j}^t - \frac{\|Y_\alpha^t\|^2}{2} - \frac{|Y_\alpha|}{2}, \quad (4.22)$$

leads to the following relation

$$N_{Y_1 Y_2}(Q; t^{-1}, q^{-1}) = (-Q)^{|\vec{Y}|} t^{\frac{\|Y_1^t\|^2 - \|Y_2^t\|^2 + |Y_1| + |Y_2|}{2}} q^{\frac{-\|Y_1\|^2 + \|Y_2\|^2 - |Y_1| - |Y_2|}{2}} N_{Y_1 Y_2}(Q^{-1}; t, q). \quad (4.23)$$

This identity immediately implies the relation $Z^{(n)}(Q, t^{-1}, q^{-1}) = Z^{(n)}(Q, t, q)$.

SQCD with fundamental flavors

The fundamental hypermultiplet gives the following additional contribution:

$$Z_{\text{fund}}^{\vec{Y}}(Q_\alpha, Q_m; t, q) = \prod_{\alpha} \prod_{(i,j) \in Y_\alpha} 2 \sinh \frac{R(a_\alpha + m + \epsilon_1(j - \frac{1}{2})\epsilon_2(i - \frac{1}{2}))}{2}, \quad (4.24)$$

To compare it with topological string partition functions, we introduce the following combinatorial factor

$$H^{\vec{Y}}(Q_\alpha, Q_m; t, q) = \prod_{\alpha} \prod_{(i,j) \in Y_\alpha} \left(1 - Q_m^{-1} Q_\alpha^{-1} t^{-i + \frac{1}{2}} q^{j - \frac{1}{2}}\right) = N_{12}^{(Y_1, \emptyset)} \left(Q_m^{-1} Q_\alpha^{-1} \sqrt{\frac{t}{q}}; t, q\right), \quad (4.25)$$

where $Q_\alpha = e^{-Ra_\alpha}$ and $Q_m = e^{-Rm}$. We then find

$$Z_{\text{fund}}^{\vec{Y}}(Q_\alpha, Q_m; t, q) = (-\sqrt{Q_m})^{|\vec{Y}|} Z_{\text{CS}}^{\vec{Y}, m = -\frac{1}{2}} H^{\vec{Y}}(Q_\alpha, Q_m; t, q), \quad (4.26)$$

where the Chern-Simons contribution is given by

$$Z_{\text{CS}}^{\vec{Y}, m} = \prod_{\alpha} Q_\alpha^{-m|Y_\alpha|} t^{-\frac{m\|Y_\alpha^t\|^2}{2}} q^{\frac{m\|Y_\alpha\|^2}{2}}. \quad (4.27)$$

Formula

$$\begin{aligned}
Z_{\text{vec}}^{\vec{Y}} &= (-1)^{N|\vec{Y}|} \mathfrak{C}_{\vec{Y}}(Q_\alpha) \prod_{\alpha} t^{\frac{(2\alpha-N)\|Y_\alpha^t\|^2}{2}} q^{\frac{(N+2-2\alpha)\|Y_\alpha\|^2}{2}} \tilde{Z}_{Y_\alpha}(t, q) \tilde{Z}_{Y_\alpha^t}(q, t) \\
&\times \prod_{1 \leq \alpha < \beta \leq N} \prod_{i, j=1}^{\infty} \frac{(1 - Q_{\beta\alpha} q^j t^{i-1})(1 - Q_{\beta\alpha} q^{j-1} t^i)}{(1 - Q_{\beta\alpha} q^{-Y_{\beta,i}+j} t^{-Y_{\alpha,j}^t+i-1})(1 - Q_{\beta\alpha} q^{-Y_{\beta,i}+j-1} t^{-Y_{\alpha,j}^t+i})}, \quad (4.28)
\end{aligned}$$

$$\mathfrak{C}_{\vec{Y}}(Q_\alpha) = \prod_{\alpha=1}^{N-1} (Q_{\alpha, \alpha+1})^{-(N-\alpha) \sum_{\beta=1}^{\alpha} |Y_\beta| - \alpha \sum_{\beta=\alpha+1}^N |Y_\beta|}. \quad (4.29)$$

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